Déduction automatique

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Je remercie les participants du mattermost qui ont eu la patience de m'expliquer mes erreurs et mes incompréhensions.
Je remercie particulièrement Jean Starynkévitch dont la lecture attentive a permis de signaler en les corrigeant de nombreuses erreurs. Dans la lancée de la ré-écriture, il m'a permis d'en repérer d'autres.
La dernière version du document est téléchargeable.
Le code source est disponible sur le git de l'UPS

Contents

Quelques Remarques

- Ceci est un ensemble d'exercices destiné à s'exercer sur l'écriture de preuves en déduction naturelle. Il ne s'agit pas d'un ensemble d'exercices sur le chapitre de la déduction naturelle. On peut le voir comme un catalogue de samples où puiser (les sources sont accessibles) plutôt qu'une œuvre musicale.
- Les sections 2 à 4 concernent les preuves dans le cadre de la logique minimale qui semble être celui du programme d'option informatique en MP.
- La section 5 concerne les preuves dans le cadre de la logique intuitionniste ; on ajoute la règle *Ex falso sequitur quod libet* qui permet de déduire toute formule d'une contradiction. On peut lire le programme en incluant cette règle sous la forme d'une modification de l'élimination de la négation qui deviendrait

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \neg\varphi}{\Gamma \vdash \psi}\ \neg_e$$

- La section 6 concerne les preuves dans le cadre de la logique classique ; on ajoute de raisonnement par l'absurde. On peut y démontrer les réciproques de preuves déjà vues.

  Sans en faire une démonstration générale, on illustre le fait qu'on passe de la logique minimale à la logique classique

  1. soit avec la règle du raisonnement par l'absurde
  2. soit par la règle de la double négation
  3. soit par la règle du tiers exclu associée à la règle de l'absurdité intuitionniste.
- Lorsque la démonstration prenait trop de place en largeur, j'ai utilisé une abréviation. Si celle-ci n'est pas indispensable pour ce motif, j'ai préféré laissé les séquents au prix d'une présentation un peu lourde.
- De même, pour des calculs trop longs, certaines déductions, 4.1.1.2, 6.1.6, 7.4.5, 7.4.6, 7.4.7, 7.4.8, sont découpées en étapes
- Il y a enfin quelques longues déductions dans le cas d'une élimination de $\vee$, 4.1.2.1, 4.3.2.2, 4.3.3.2, 4.4.3.2, 7.2.4.2, pour lesquelles la preuve de la seconde branche n'a pas été écrite.
- J'introduit une règle de renommage, $\dfrac{\Gamma \vdash \varphi}{\Gamma \vdash \psi}\ \rho$, qui permet de remplacer φ par ψ quand ces deux formules désignent la même chose.
- Il y a des erreurs dans ce document, n'hésitez pas à me les signaler, je corrigerai celles-ci.

1 Règles de la déduction naturelle

1.1 Logique propositionnelle minimale

- Axiome, hypothèse ou réflexivité $\dfrac{}{\Gamma, \varphi \vdash \varphi}\ \mathrm{ax}$
- Monotonie ou affaiblissement $\dfrac{\Gamma \vdash \varphi}{\Gamma, \Delta \vdash \varphi}\ \mathrm{aff}$
- Introduction de la conjonction $\dfrac{\Gamma \vdash \varphi_1 \quad \Gamma \vdash \varphi_2}{\Gamma \vdash \varphi_1 \wedge \varphi_2}\ \wedge_i$
- Élimination de la conjonction $\dfrac{\Gamma \vdash \varphi_1 \wedge \varphi_2}{\Gamma \vdash \varphi_1}\ \wedge_e \qquad \dfrac{\Gamma \vdash \varphi_1 \wedge \varphi_2}{\Gamma \vdash \varphi_2}\ \wedge_e$
- Introduction de la disjonction $\dfrac{\Gamma \vdash \varphi_1}{\Gamma \vdash \varphi_1 \vee \varphi_2}\ \vee_i \qquad \dfrac{\Gamma \vdash \varphi_2}{\Gamma \vdash \varphi_1 \vee \varphi_2}\ \vee_i$
- Élimination de la disjonction $\dfrac{\Gamma \vdash \varphi_1 \vee \varphi_2 \quad \Gamma, \varphi_1 \vdash \psi \quad \Gamma, \varphi_2 \vdash \psi}{\Gamma \vdash \psi}\ \vee_e$
- Introduction de l'implication $\dfrac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}\ \to_i$
- Élimination de l'implication ou `modus ponens` $\dfrac{\Gamma \vdash \varphi \to \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi}\ \to_e$
- Introduction de la négation $\dfrac{\Gamma, \varphi \vdash \perp}{\Gamma \vdash \neg\varphi}\ \neg_i$
- Élimination de la négation $\dfrac{\Gamma \vdash \varphi \quad \Gamma \vdash \neg\varphi}{\Gamma \vdash \perp}\ \neg_e$

1.2 Logique propositionnelle intuitionniste

- Absurdité intuitioniste $\dfrac{\Gamma \vdash \perp}{\Gamma \vdash \psi}\ \perp_e$

  Son emploi ne se fait que dans le cadre de la logique intuitioniste.

1.3 Raisonnement par l'absurde

On énonce 3 règles qui sont équivalentes.
Leur emploi ne se fait que d ans le cadre de la logique classique.

- Tiers exclu $\dfrac{}{\Gamma \vdash \varphi \vee \neg\varphi}\ \mathrm{te}$
- Élimination de la double négation $\dfrac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}\ \neg\neg_e$
- Raisonnement par l'absurde $\dfrac{\Gamma, \neg\varphi \vdash \perp}{\Gamma \vdash \varphi}\ \mathrm{raa}$

1.4 Logique du premier ordre

La notation $\varphi[x \leftarrow t]$ signifie qu'on a remplacé la variable x par le terme t dans φ.

- Introduction du quantificateur universel $\dfrac{\Gamma \vdash \varphi \qquad x \text{ n'est pas une variable libre de } \Gamma}{\Gamma \vdash \forall x \varphi}\ \forall_i$
- Élimination du quantificateur universel $\dfrac{\Gamma \vdash \forall x \varphi}{\Gamma \vdash \varphi[x \leftarrow t]}\ \forall_e$

  pour tout terme t dont les variables libres ne sont pas des variables liées dans φ. En utilisant la substitution $\varphi[x \leftarrow x]$ qui redonne φ, cette règle sera souvent employée sous la forme $\dfrac{\Gamma \vdash \forall x \varphi}{\Gamma \vdash \varphi}\ \forall_e$
- Introduction du quantificateur existentiel $\dfrac{\Gamma \vdash \varphi[x \leftarrow t]}{\Gamma \vdash \exists x \varphi}\ \exists_i$ En remarquant qu'on peut écrire φ sous la forme $\varphi[x \leftarrow x]$, cette règle sera souvent employée sous la forme $\dfrac{\Gamma \vdash \varphi}{\Gamma \vdash \exists x \varphi}\ \exists_i$
- Élimination du quantificateur existentiel $\dfrac{\Gamma \vdash \exists x \varphi \qquad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi}\ \exists_e$

  si x n'est pas une variable libre de Γ ni de ψ.

2 Exercices simples

2.1 Résultats naturels

2.1.1 $\vdash p \to p$

$$\dfrac{\dfrac{}{p \vdash p}\,\text{ax}}{\vdash p \to p}\,\to_i$$

2.1.2 $p, \neg p \vdash \bot$

$$\dfrac{\dfrac{}{p, \neg p \vdash p}\,\text{ax} \qquad \dfrac{}{p, \neg p \vdash \neg p}\,\text{ax}}{p, \neg p \vdash \bot}\,\neg_e$$

2.1.3 $p, q \vdash p \wedge q$

$$\dfrac{\dfrac{}{p, q \vdash p}\,\text{ax} \qquad \dfrac{}{p, q \vdash q}\,\text{ax}}{p, q \vdash p \wedge q}\,\wedge_i$$

2.1.4 $p \wedge q \vdash q \wedge p$

$$\dfrac{\dfrac{\dfrac{}{p \wedge q \vdash p \wedge q}\,\text{ax}}{p \wedge q \vdash q}\,\wedge_e \qquad \dfrac{\dfrac{}{p \wedge q \vdash p \wedge q}\,\text{ax}}{p \wedge q \vdash p}\,\wedge_e}{p \wedge q \vdash q \wedge p}\,\wedge_i$$

2.1.5 $p \vee q \vdash q \vee p$

$$\dfrac{\dfrac{}{p \vee q \vdash p \vee q}\,\text{ax} \qquad \dfrac{\dfrac{}{p \vee q, p \vdash p}\,\text{ax}}{p \vee q, p \vdash q \vee p}\,\vee_i \qquad \dfrac{\dfrac{}{p \vee q, q \vdash q}\,\text{ax}}{p \vee q, q \vdash q \vee p}\,\vee_i}{p \vee q \vdash q \vee p}\,\vee_e$$

2.1.6 $\neg\neg\neg p \vdash \neg p$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\neg\neg p, p, \neg p \vdash p}\,\text{ax} \qquad \dfrac{}{\neg\neg\neg p, p, \neg p \vdash \neg p}\,\text{ax}}{\neg\neg\neg p, p, \neg p \vdash \bot}\,\neg_e}{\neg\neg\neg p, p \vdash \neg(\neg p)}\,\neg_i \qquad \dfrac{}{\neg\neg\neg p, p \vdash \neg(\neg\neg p)}\,\text{ax}}{\neg\neg\neg p, p \vdash \bot}\,\neg_e}{\neg\neg\neg p \vdash \neg p}\,\neg_i$$

2.2 Implications

2.2.1 $q \vdash p \to q$

$$\dfrac{\dfrac{}{q, p \vdash q}\ \text{ax}}{q \vdash p \to q}\ \to_i$$

2.2.2 $p \wedge q \vdash p \to q$

$$\dfrac{\dfrac{\dfrac{}{p \wedge q, p \vdash p \wedge q}\ \text{ax}}{p \wedge q, p \vdash q}\ \wedge_e}{p \wedge q \vdash p \to q}\ \to_i$$

2.2.3 $p \to q \vdash p \to (p \wedge q)$

$$\dfrac{\dfrac{\dfrac{}{p \to q, p \vdash p}\ \text{ax} \qquad \dfrac{\dfrac{}{p \to q, p \vdash p \to q}\ \text{ax} \qquad \dfrac{}{p \to q, p \vdash p}\ \text{ax}}{p \to q, p \vdash q}\ \to_e}{p \to q, p \vdash p \wedge q}\ \wedge_i}{p \to q \vdash p \to (p \wedge q)}\ \to_i$$

2.2.4 $p \to r \vdash (p \wedge q) \to r$

$$\dfrac{\dfrac{\dfrac{}{p \to r, p \wedge q \vdash p \to r}\ \text{ax} \qquad \dfrac{\dfrac{}{p \to r, p \wedge q \vdash p \wedge q}\ \text{ax}}{p \to r, q \to r, p \wedge q \vdash p}\ \wedge_e}{p \to r, p \wedge q \vdash r}\ \to_e}{p \to r \vdash (p \wedge q) \to r}\ \to_i$$

2.2.5 $\vdash p \to (q \to p)$

$$\dfrac{\dfrac{\dfrac{}{p, q \vdash p}\ \text{ax}}{p \vdash q \to p}\ \to_i}{\vdash p \to (q \to p)}\ \to_i$$

2.2.6 $p \vdash (p \to q) \to q$

$$\dfrac{\dfrac{\dfrac{}{p, p \to q \vdash p \to q}\ \text{ax} \qquad \dfrac{}{p, p \to q \vdash p}\ \text{ax}}{p, p \to q \vdash q}\ \to_e}{p \vdash (p \to q) \to q}\ \to_i$$

2.3 Divers

2.3.1 $p \vee (p \wedge q) \vdash p$

$$\dfrac{\dfrac{}{p \vee (p \wedge q) \vdash p \vee (p \wedge q)}\,\text{ax} \qquad \dfrac{}{p \vee (p \wedge q), p \vdash p}\,\text{ax} \qquad \dfrac{\dfrac{}{p \vee (p \wedge q), p \wedge q \vdash p \wedge q}\,\text{ax}}{p \vee (p \wedge q), p \wedge q \vdash p}\,\wedge_e}{p \vee (p \wedge q) \vdash p}\,\vee_e$$

2.3.2 $p \wedge q, r \wedge s \vdash p \wedge s$

$$\dfrac{\dfrac{\dfrac{}{p \wedge q, r \wedge s \vdash p \wedge q}\,\text{ax}}{p \wedge q, r \wedge s \vdash p}\,\wedge_e \qquad \dfrac{\dfrac{}{p \wedge q, r \wedge s \vdash r \wedge s}\,\text{ax}}{p \wedge q, r \wedge s \vdash s}\,\wedge_e}{p \wedge q, r \wedge s \vdash p \wedge s}\,\wedge_i$$

2.3.3 $p, q \wedge r \vdash p \wedge q$

$$\dfrac{\dfrac{}{p, q \wedge r \vdash p}\,\text{ax} \qquad \dfrac{\dfrac{}{p, q \wedge r \vdash q \wedge r}\,\text{ax}}{p, q \wedge r \vdash q}\,\wedge_e}{p, q \wedge r \vdash p \wedge q}\,\wedge_i$$

2.3.4 $p \vdash \neg\neg p$

$$\dfrac{\dfrac{\dfrac{}{p, \neg p \vdash p}\,\text{ax} \qquad \dfrac{}{p, \neg p \vdash \neg p}\,\text{ax}}{p, \neg p \vdash \bot}\,\neg_e}{p \vdash \neg\neg p}\,\neg_i$$

2.3.5 $\vdash \neg(p \wedge \neg p)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \wedge \neg p \vdash p \wedge \neg p}\,\text{ax}}{p \wedge \neg p \vdash p}\,\wedge_e \qquad \dfrac{\dfrac{}{p \wedge \neg p \vdash p \wedge \neg p}\,\text{ax}}{p \wedge \neg p \vdash \neg p}\,\wedge_e}{p \wedge \neg p \vdash \bot}\,\neg_e}{\vdash \neg(p \wedge \neg p)}\,\neg_i$$

3 Implications

3.1 Simplifications

3.1.1 $p \to \neg p \vdash \neg p$

$$\dfrac{\dfrac{\dfrac{}{p \to \neg p, p \vdash p}\,\mathrm{ax} \qquad \dfrac{\dfrac{}{p \to \neg p, p \vdash p \to \neg p}\,\mathrm{ax} \quad \dfrac{}{p \to \neg p, p \vdash p}\,\mathrm{ax}}{p \to \neg p, p \vdash \neg p}\,\to_e}{p \to \neg p, p \vdash \bot}\,\neg_e}{p \to \neg p \vdash \neg p}\,\neg_i$$

La forme $\neg p \to p \vdash p$ se démontre en logique classique, voir 6.1.1.

3.1.2 $p \to q, \neg q \vdash \neg p$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \to q, \neg q, p \vdash p \to q}\,\mathrm{ax} \quad \dfrac{}{p \to q, \neg q, p \vdash p}\,\mathrm{ax}}{p \to q, \neg q, p \vdash q}\,\to_e \qquad \dfrac{}{p \to q, \neg q, p \vdash \neg q}\,\mathrm{ax}}{p \to q, \neg q, p \vdash \bot}\,\neg_e}{p \to q, \neg q \vdash \neg p}\,\neg_i$$

3.1.3 $p \to q, p \vee q \vdash q$

On note $\Gamma = p \to q, p \vee q$.

$$\dfrac{\dfrac{}{p \to q, p \vee q \vdash p \vee q}\,\mathrm{ax} \qquad \dfrac{\dfrac{}{p \to q, p \vee q, p \vdash p \to q}\,\mathrm{ax} \quad \dfrac{}{p \to q, p \vee q, p \vdash p}\,\mathrm{ax}}{p \to q, p \vee q, p \vdash q}\,\to_e \qquad \dfrac{}{p \to q, p \vee q, q \vdash q}\,\mathrm{ax}}{p \to q, p \vee q \vdash q}\,\vee_e$$

3.1.4 $p \to q, p \to \neg q \vdash \neg p$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \to q, p \to \neg q, p \vdash p \to q}\,\mathrm{ax} \quad \dfrac{}{p \to q, p \to \neg q, p \vdash p}\,\mathrm{ax}}{p \to q, p \to \neg q, p \vdash q}\,\to_e \qquad \dfrac{\dfrac{}{p \to q, p \to \neg q, p \vdash p \to \neg q}\,\mathrm{ax} \quad \dfrac{}{p \to q, p \to \neg q, p \vdash p}\,\mathrm{ax}}{p \to q, p \to \neg q, p \vdash \neg q}\,\to_e}{p \to q, p \to \neg q, p \vdash \bot}\,\neg_e}{p \to q, p \to \neg q \vdash \neg p}\,\neg_i$$

3.1.5 $p \to (q \vee r), \neg q, \neg r \vdash \neg p$

On note $\Gamma = p \to (q \vee r), \neg q, \neg r, p$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash p \to (q \vee r)}\,\mathrm{ax} \quad \dfrac{}{\Gamma \vdash p}\,\mathrm{ax}}{\Gamma \vdash q \vee r}\,\to_e \qquad \dfrac{\dfrac{}{\Gamma, q \vdash q}\,\mathrm{ax} \quad \dfrac{}{\Gamma, q \vdash \neg q}\,\mathrm{ax}}{\Gamma, q \vdash \bot}\,\to_e \qquad \dfrac{\dfrac{}{\Gamma, r \vdash r}\,\mathrm{ax} \quad \dfrac{}{\Gamma, r \vdash \neg r}\,\mathrm{ax}}{\Gamma, r \vdash \bot}\,\to_e}{p \to (q \vee r), \neg q, \neg r, p \vdash \bot}\,\vee_e}{p \to (q \vee r), \neg q, \neg r \vdash \neg p}\,\neg_i$$

3.1.6 $p \to (q \to r), p, \neg r \vdash \neg q$

On note $\Gamma = p \to (q \to r), p, \neg r, q$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash p \to (q \to r)}\,\mathrm{ax} \quad \dfrac{}{\Gamma \vdash p}\,\mathrm{ax}}{\Gamma \vdash q \to r}\,\to_e \qquad \dfrac{}{p \to (q \to r), p, \neg r, q \vdash q}\,\mathrm{ax}}{p \to (q \to r), p, \neg r, q \vdash r}\,\to_e \qquad \dfrac{}{p \to (q \to r), p, \neg r, q \vdash \neg r}\,\mathrm{ax}}{p \to (q \to r), p, \neg r, q \vdash \bot}\,\neg_e}{p \to (q \to r), p, \neg r \vdash \neg q}\,\neg_i$$

3.1.7 $p \to (q \to r), p \to q, p \vdash r$

On note $\Gamma = p \to (q \to r), p \to q, p$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash p \to (q \to r)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash p$}
\RightLabel{$\to_e$}
\BinaryInfC{$p \to (q \to r), p \to q, p \vdash q \to r$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash p \to q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash p$}
\RightLabel{$\to_e$}
\BinaryInfC{$p \to (q \to r), p \to q, p \vdash q$}
\RightLabel{$\to_e$}
\BinaryInfC{$p \to (q \to r), p \to q, p \vdash r$}
\end{prooftree}
$$

3.1.8 $p \to (q \to r), q \to p \vdash q \to r$

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\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p \to (q \to r), q \vdash p \to (q \to r)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p \to (q \to r) q \vdash q \to p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p \to (q \to r), q \vdash q$}
\RightLabel{$\to_e$}
\BinaryInfC{$p \to (q \to r), q \vdash p$}
\RightLabel{$\to_e$}
\BinaryInfC{$p \to (q \to r), q \vdash q \to r$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p \to (q \to r), q \vdash q$}
\RightLabel{$\to_e$}
\BinaryInfC{$p \to (q \to r), q \vdash r$}
\RightLabel{$\to_i$}
\UnaryInfC{$p \to (q \to r), q \to r \vdash q \to r$}
\end{prooftree}
$$

3.1.9 $p \to (p \to q), p \vdash q$

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\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p \to (p \to q), p \vdash p \to (p \to q)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p \to (p \to q), p \vdash p$}
\RightLabel{$\to_e$}
\BinaryInfC{$p \to (p \to q), p \vdash p \to q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p \to (p \to q), p \vdash p$}
\RightLabel{$\to_e$}
\BinaryInfC{$p \to (p \to q), p \vdash q$}
\end{prooftree}
$$

3.2 Transformations

3.2.1 $p \to q \vdash \neg q \to \neg p$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{p \to q, \neg q, p \vdash p \to q}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{p \to q, \neg q, p \vdash p}
\RightLabel{\to_e}
\BinaryInfC{p \to q, \neg q, p \vdash q}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{p \to q, \neg q, p \vdash \neg q}
\RightLabel{\neg_e}
\BinaryInfC{p \to q, \neg q, p \vdash \bot}
\RightLabel{\neg_i}
\UnaryInfC{p \to q, \neg q \vdash \neg p}
\RightLabel{\to_i}
\UnaryInfC{p \to q \vdash \neg q \to \neg p}
\end{prooftree}
$$

C'est la principe de la transposition.
La réciproque se démontre dans le cadre de la logique classique, voir l'exercice 6.1.4.

3.2.2 $p \to q \vdash (p \wedge r) \to (q \wedge r)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{p \to q, p \wedge r \vdash p \to q}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{p \to q, p \wedge r \vdash p \wedge r}
\RightLabel{\wedge_e}
\UnaryInfC{p \to q, p \wedge r \vdash p}
\RightLabel{\to_e}
\BinaryInfC{p \to q, p \wedge r \vdash q}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{p \to q, p \wedge r \vdash p \wedge r}
\RightLabel{\wedge_e}
\UnaryInfC{p \to q, p \wedge r \vdash r}
\RightLabel{\wedge_i}
\BinaryInfC{p \to q, p \wedge r \vdash q \wedge r}
\RightLabel{\to_i}
\UnaryInfC{p \to q \vdash (p \wedge r) \to (q \wedge r)}
\end{prooftree}
$$

3.2.3 $(p \wedge r) \to (q \wedge r), r \vdash p \to q$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{(p \wedge r) \to (q \wedge r), r, p \vdash (p \wedge r) \to (q \wedge r)}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{(p \wedge r) \to (q \wedge r), r, p \vdash p}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{(p \wedge r) \to (q \wedge r), r, p \vdash r}
\RightLabel{\wedge_i}
\BinaryInfC{(p \wedge r) \to (q \wedge r), r, p \vdash p \wedge r}
\RightLabel{\to_e}
\BinaryInfC{(p \wedge r) \to (q \wedge r), r, p \vdash q \wedge r}
\RightLabel{\wedge_e}
\UnaryInfC{(p \wedge r) \to (q \wedge r), r, p \vdash q}
\RightLabel{\to_i}
\UnaryInfC{(p \wedge r) \to (q \wedge r), r \vdash p \to q}
\end{prooftree}
$$

3.2.4 $p \to q \vdash (p \vee r) \to (q \vee r)$

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\begin{prooftree}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{p \to q, p \vee r \vdash p \vee r}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{p \to q, p \vee r, p \vdash p \to q}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{p \to q, p \vee r, p \vdash p}
\RightLabel{\to_e}
\BinaryInfC{p \to q, p \vee r, p \vdash q}
\RightLabel{\vee_i}
\UnaryInfC{p \to q, p \vee r, p \vdash q \vee r}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{p \to q, p \vee r, p \vdash r}
\RightLabel{\vee_i}
\UnaryInfC{p \to q, p \vee r, q \vdash q \vee r}
\RightLabel{\vee_e}
\TrinaryInfC{p \to q, p \vee r \vdash q \vee r}
\RightLabel{\to_i}
\UnaryInfC{p \to q \vdash (p \vee r) \to (q \vee r)}
\end{prooftree}
$$

On peut prouver $(p \vee r) \to (q \vee r), \neg r \vdash p \to q$ mais on a besoin de $(\bot_e)$, voir l'exercice 5.1.4

3.2.5 $(p \wedge q) \to r, \neg r \vdash p \to \neg q$

On note $\Gamma = (p \wedge q) \to r, \neg r, p, q$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{(p \wedge q) \to r, \neg r, p, q \vdash (p \wedge q) \to r}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{\Gamma \vdash p}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{\Gamma \vdash q}
\RightLabel{\wedge_i}
\BinaryInfC{(p \wedge q) \to r, \neg r, p, q \vdash p \wedge q}
\RightLabel{\to_e}
\BinaryInfC{(p \wedge q) \to r, \neg r, p, q \vdash r}
\AxiomC{}
\RightLabel{\text{ax}}
\UnaryInfC{(p \wedge q) \to r, \neg r, p, q \vdash \neg r}
\RightLabel{\neg_e}
\BinaryInfC{(p \wedge q) \to r, \neg r, p, q \vdash \bot}
\RightLabel{\neg_i}
\UnaryInfC{(p \wedge q) \to r, \neg r, p \vdash \neg q}
\RightLabel{\to_i}
\UnaryInfC{(p \wedge q) \to r, \neg r \vdash p \to \neg q}
\end{prooftree}
$$

3.2.6 $p \to q, r \to s \vdash (p \wedge r) \to (q \wedge s)$

On note $\Gamma = p \to q, r \to s, p \wedge r$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash p \to q}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma \vdash p \wedge r}\,\text{ax}}{\Gamma \vdash p}\,\wedge_e}{p \to q, r \to s, p \wedge r \vdash q}\,\to_e \qquad \dfrac{\dfrac{}{\Gamma \vdash r \to s}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma \vdash p \wedge r}\,\text{ax}}{\Gamma \vdash r}\,\wedge_e}{p \to q, r \to s, p \wedge r \vdash s}\,\to_e}{p \to q, r \to s, p \wedge r \vdash q \wedge s}\,\wedge_i}{p \to q, r \to s \vdash (p \wedge r) \to (q \wedge s)}\,\to_i$$

3.2.7 $p \to q, r \to s \vdash (p \vee r) \to (q \vee s)$

On note $\Gamma = p \to q, r \to s, p \vee r$.

$$\dfrac{\dfrac{\dfrac{}{p \to q, r \to s, p \vee r \vdash p \vee r}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, p \vdash p \to q}\,\text{ax} \qquad \dfrac{}{\Gamma, p \vdash p}\,\text{ax}}{p \to q, r \to s, p \vee r, p \vdash q}\,\to_e}{p \to q, r \to s, p \vee r, p \vdash q \vee s}\,\vee_i \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, r \vdash r \to s}\,\text{ax} \qquad \dfrac{}{\Gamma, r \vdash r}\,\text{ax}}{p \to q, r \to s, p \vee r, r \vdash s}\,\to_e}{p \to q, r \to s, p \vee r, r \vdash q \vee s}\,\vee_i}{p \to q, r \to s, p \vee r \vdash q \vee s}\,\vee_e}{p \to q, r \to s \vdash (p \vee r) \to (q \vee s)}\,\to_i$$

3.2.8 $p \to (q \vee r), q \to s, r \to s \vdash p \to s$

On note $\Gamma = p \to (q \vee r), q \to s, r \to s, p$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash p \to (q \vee r)}\,\text{ax} \qquad \dfrac{}{\Gamma \vdash p}\,\text{ax}}{\Gamma \vdash q \vee r}\,\to_e \qquad \dfrac{\dfrac{}{\Gamma, q \vdash q \to s}\,\text{ax} \qquad \dfrac{}{\Gamma, q \vdash q}\,\text{ax}}{\Gamma, q \vdash s}\,\to_e \qquad \dfrac{\dfrac{}{\Gamma, r \vdash r \to s}\,\text{ax} \qquad \dfrac{}{\Gamma, r \vdash r}\,\text{ax}}{\Gamma, r \vdash s}\,\to_e}{p \to (q \to r), q \to s, r \to s, p \vdash s}\,\vee_e}{p \to (q \to r), q \to s, r \to s \vdash p \to s}\,\to_i$$

3.2.9 $p \to (q \to r) \vdash q \to (p \to r)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \to (q \to r), p, q \vdash p \to (q \to r)}\,\text{ax} \qquad \dfrac{}{p \to (q \to r), p, q \vdash p}\,\text{ax}}{p \to (q \to r), p, q \vdash q \to r}\,\to_e \qquad \dfrac{}{p \to (q \to r), p, q \vdash q}\,\text{ax}}{p \to (q \to r), p, q \vdash r}\,\to_e}{p \to (q \to r), q \vdash p \to r}\,\to_i}{p \to (q \to r) \vdash q \to (p \to r)}\,\to_i$$

4 Équivalences classiques

4.1 Distributivités

4.1.1 $p \land (q \lor r)$ et $(p \land q) \lor (p \land r)$

4.1.1.1 $p \land (q \lor r) \vdash (p \land q) \lor (p \land r)$

On note $\varphi = p \land (q \lor r)$.

$$\dfrac{\dfrac{\dfrac{}{\varphi \vdash p \land (q \lor r)}\,\text{ax}}{\varphi \vdash q \lor r}\,\land_e \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi, q \vdash p \land (q \lor r)}\,\text{ax}}{\varphi, q \vdash p}\,\land_e \quad \dfrac{}{\varphi, q \vdash q}\,\text{ax}}{\varphi, q \vdash p \land q}\,\land_i}{\varphi, q \vdash (p \land q) \lor (p \land r)}\,\lor_i \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi, r \vdash p \land (q \lor r)}\,\text{ax}}{\varphi, r \vdash p}\,\land_e \quad \dfrac{}{\varphi, r \vdash r}\,\text{ax}}{\varphi, r \vdash p \land r}\,\land_i}{\varphi, r \vdash (p \land q) \lor (p \land r)}\,\lor_i}{p \land (q \lor r) \vdash (p \land q) \lor (p \land r)}\,\lor_e$$

4.1.1.2 $(p \land q) \lor (p \land r) \vdash p \land (q \lor r)$

Lemme 1 On commence par prouver $(p \land q) \lor (p \land r) \vdash p$.

$$\dfrac{\dfrac{}{(p \land q) \lor (p \land r) \vdash \varphi}\,\text{ax} \quad \dfrac{\dfrac{}{(p \land q) \lor (p \land r), p \land q \vdash p \land q}\,\text{ax}}{(p \land q) \lor (p \land r), p \land q \vdash p}\,\land_e \quad \dfrac{\dfrac{}{(p \land q) \lor (p \land r), p \land r \vdash p \land r}\,\text{ax}}{(p \land q) \lor (p \land r), p \land r \vdash p}\,\land_e}{(p \land q) \lor (p \land r) \vdash p}\,\lor_e$$

Lemme 2 On prouve ensuite $(p \land q) \lor (p \land r) \vdash q \lor r$.

$$\dfrac{\dfrac{}{(p \land q) \lor (p \land r) \vdash (p \land q) \lor (p \land r)}\,\text{ax} \quad \dfrac{\dfrac{\dfrac{}{(p \land q) \lor (p \land r), p \land q \vdash p \land q}\,\text{ax}}{q \lor r, p \land q \vdash q}\,\land_e}{(p \land q) \lor (p \land r) p \land q \vdash q \lor r}\,\lor_i \quad \dfrac{\dfrac{\dfrac{}{(p \land q) \lor (p \land r), p \land r \vdash p \land r}\,\text{ax}}{(p \land q) \lor (p \land r), p \land r \vdash r}\,\land_e}{(p \land q) \lor (p \land r), p \land q \vdash q \lor r}\,\lor_i}{(p \land q) \lor (p \land r) \vdash q \lor r}\,\lor_e$$

Conclusion On assemble les deux déductions

$$\dfrac{\dfrac{\text{Lemme 1}}{(p \land q) \lor (p \land r) \vdash p} \quad \dfrac{\text{Lemme 2}}{(p \land q) \lor (p \land r) \vdash q \lor r}}{(p \land q) \lor (p \land r) \vdash p \land (q \lor r)}\,\land_i$$

4.1.2 $p \lor (q \land r)$ et $(p \lor q) \land (p \lor r)$

4.1.2.1 $p \lor (q \land r) \vdash (p \lor q) \land (p \lor r)$

$$\dfrac{\dfrac{\dfrac{}{p \lor (q \land r) \vdash p \lor (q \land r)}\,\text{ax} \quad \dfrac{\dfrac{}{p \lor (q \land r), p \vdash p}\,\text{ax}}{p \lor (q \land r), p \vdash p \lor q}\,\lor_i \quad \dfrac{\dfrac{\dfrac{}{p \lor (q \land r), q \land r \vdash q \land r}\,\text{ax}}{p \lor (q \land r), q \land r \vdash q}\,\land_e}{p \lor (q \land r), q \land r \vdash p \lor q}\,\lor_i}{p \lor (q \land r) \vdash p \lor q}\,\lor_e \quad \dfrac{\text{Idem}}{p \lor (q \land r) \vdash p \lor q}}{p \lor (q \land r) \vdash (p \lor q) \land (p \lor r)}\,\land_i$$

4.1.2.2 $(p \lor q) \land (p \lor r) \vdash p \lor (q \land r)$

On note $\varphi = (p \lor q) \land (p \lor r)$ et $\Gamma = (p \lor q) \land (p \lor r), q$.

$$\dfrac{\dfrac{\dfrac{}{\varphi \vdash (p \lor q) \land (p \lor r)}\,\text{ax}}{\varphi \vdash p \lor q}\,\land_e \quad \dfrac{\dfrac{}{\varphi, p \vdash p}\,\text{ax}}{\varphi, p \vdash p \lor (q \land r)}\,\lor_i \quad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash (p \lor q) \land (p \lor r)}\,\text{ax}}{\Gamma \vdash p \lor r}\,\land_e \quad \dfrac{\dfrac{}{\Gamma, p \vdash p}\,\text{ax}}{\Gamma, p \vdash p \lor (q \land r)}\,\lor_i \quad \dfrac{\dfrac{\dfrac{}{\Gamma, r \vdash q}\,\text{ax} \quad \dfrac{}{\Gamma, r \vdash r}\,\text{ax}}{\Gamma, r \vdash q \land r}\,\land_i}{\Gamma, r \vdash p \lor (q \land r)}\,\lor_i}{\Gamma \vdash p \lor (q \land r)}\,\lor_e}{(p \lor q) \land (p \lor r) \vdash p \lor (q \land r)}\,\lor_e$$

4.2 Lois de de Morgan

4.2.1 $\neg(p \lor q)$ et $\neg p \land \neg q$

4.2.1.1 $\neg(p \lor q) \vdash \neg p \land \neg q$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(p \lor q), p \vdash \neg(p \lor q)}\,\text{ax} \quad \dfrac{\dfrac{}{\neg(p \lor q), p \vdash p}\,\text{ax}}{\neg(p \lor q), p \vdash p \lor q}\,\lor_i}{\neg(p \lor q), p \vdash \bot}\,\neg_e}{\neg(p \lor q) \vdash \neg p}\,\neg_i \quad \dfrac{\dfrac{\dfrac{}{\neg(p \lor q), q \vdash \neg(p \lor q)}\,\text{ax} \quad \dfrac{\dfrac{}{\neg(p \lor q), q \vdash q}\,\text{ax}}{\neg(p \lor q), q \vdash p \lor q}\,\lor_i}{\neg(p \lor q), q \vdash \bot}\,\neg_e}{\neg(p \lor q) \vdash \neg q}\,\neg_i}{\neg(p \lor q) \vdash \neg p \land \neg q}\,\land_i$$

4.2.1.2 $\neg p \land \neg q \vdash \neg(p \lor q)$

On note $\varphi = \neg p \land \neg q$ et $\psi = p \lor q$.

$$\dfrac{\dfrac{\dfrac{}{\neg p \land \neg q, p \lor q \vdash p \lor q}\,\text{ax} \quad \dfrac{\dfrac{}{\varphi, \psi, p \vdash p}\,\text{ax} \quad \dfrac{\dfrac{}{\varphi, \psi, p \vdash \neg p \land \neg q}\,\text{ax}}{\varphi, \psi, p \vdash \neg p}\,\land_e}{\neg p \land \neg q, p \lor q, p \vdash \bot}\,\neg_e \quad \dfrac{\dfrac{}{\varphi, \psi, q \vdash q}\,\text{ax} \quad \dfrac{\dfrac{}{\varphi, \psi, q \vdash \neg p \land \neg q}\,\text{ax}}{\varphi, \psi, q \vdash \neg q}\,\land_e}{\neg p \land \neg q, p \lor q, q \vdash \bot}\,\neg_e}{\neg p \land \neg q, p \lor q \vdash \bot}\,\lor_e}{\neg p \land \neg q \vdash \neg(p \lor q)}\,\neg_i$$

4.2.2 $\neg(p \land q)$ et $\neg p \lor \neg q$

4.2.2.1 $\neg(p \land q) \vdash \neg p \lor \neg q$

La déduction se fait en logique classique, voir l'exercice 6.1.7

4.2.2.2 $\neg p \lor \neg q \vdash \neg(p \land q)$

On note $\varphi = \neg p \lor \neg q$ et $\psi = p \land q$.

$$\dfrac{\dfrac{\dfrac{}{\varphi, \psi \vdash \neg p \lor \neg q}\,\text{ax} \quad \dfrac{\dfrac{\dfrac{}{\varphi, \psi, \neg p \vdash p \land q}\,\text{ax}}{\varphi, \psi, \neg p \vdash p}\,\land_e \quad \dfrac{}{\varphi, \psi, \neg p \vdash \neg p}\,\text{ax}}{\varphi, \psi, \neg p \vdash \bot}\,\neg_e \quad \dfrac{\dfrac{\dfrac{}{\varphi, \psi, \neg q \vdash p \land q}\,\text{ax}}{\varphi, \psi, \neg q \vdash q}\,\land_e \quad \dfrac{}{\varphi, \psi, \neg q \vdash \neg q}\,\text{ax}}{\varphi, \psi, \neg q \vdash \bot}\,\neg_e}{\neg p \lor \neg q, p \land q \vdash \bot}\,\lor_e}{\neg p \lor \neg q \vdash \neg(p \land q)}\,\neg_i$$

4.3 3 formules équivalentes

4.3.1 $(p \wedge q) \to r$ et $p \to (q \to r)$

4.3.1.1 $(p \wedge q) \to r \vdash p \to (q \to r)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(p \wedge q) \to r, p, q \vdash (p \wedge q) \to r}\,\text{ax} \qquad \dfrac{\dfrac{}{(p \wedge q) \to r, p, q \vdash p}\,\text{ax} \qquad \dfrac{}{(p \wedge q) \to r, p, q \vdash q}\,\text{ax}}{(p \wedge q) \to r, p, q \vdash p \wedge q}\,\wedge_i}{(p \wedge q) \to r, p, q \vdash r}\,\to_e}{(p \wedge q) \to r, p \vdash q \to r}\,\to_i}{(p \wedge q) \to r \vdash p \to (q \to r)}\,\to_i$$

4.3.1.2 $p \to (q \to r) \vdash (p \wedge q) \to r$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \to (q \to r), p \wedge q \vdash p \to (q \to r)}\,\text{ax} \qquad \dfrac{\dfrac{}{p \to (q \to r), p \wedge q \vdash p \wedge q}\,\text{ax}}{p \to (q \to r), p \wedge q \vdash p}\,\wedge_e}{p \to (q \to r), p \wedge q \vdash q \to r}\,\to_e \qquad \dfrac{\dfrac{}{p \to (q \to r), p \wedge q \vdash p \wedge q}\,\text{ax}}{p \to (q \to r), p \wedge q \vdash q}\,\wedge_e}{p \to (q \to r), p \wedge q \vdash r}\,\to_e}{p \to (q \to r) \vdash (p \wedge q) \to r}\,\to_i$$

4.3.2 $(p \wedge q) \to r$ et $(p \to r) \vee (q \to r)$

4.3.2.1 $(p \wedge q) \to r \vdash (p \to r) \vee (q \to r)$

Ce résultat se déduit dans la logique classique, voir l'exercice 6.1.8.

4.3.2.2 $(p \to r) \vee (q \to r) \vdash (p \wedge q) \to r$

On note $\Gamma = (p \to r) \vee (q \to r), p \wedge q$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (p \to r) \vee (q \to r)}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma, p \to r \vdash p \to r}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma, p \to r \vdash p \wedge q}\,\text{ax}}{\Gamma, p \to r \vdash p}\,\wedge_e}{\Gamma, p \to r \vdash r}\,\to_e \qquad \dfrac{\text{Idem}}{\Gamma, q \to r \vdash r}}{(p \to r) \vee (q \to r), p \wedge q \vdash r}\,\vee_e}{(p \to r) \vee (q \to r) \vdash (p \wedge q) \to r}\,\to_i$$

4.3.3 $p \to (q \to r)$ et $(p \to r) \vee (q \to r)$

4.3.3.1 $p \to (q \to r) \vdash (p \to r) \vee (q \to r)$

Ce résultat se déduit dans la logique classique, voir l'exercice 6.1.9.

4.3.3.2 $(p \to r) \vee (q \to r) \vdash p \to (q \to r)$

On note $\Gamma = (p \to r) \vee (q \to r), p, q$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (p \to r) \vee (q \to r)}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma, p \to r \vdash p \to r}\,\text{ax} \qquad \dfrac{}{\Gamma, p \to r \vdash p}\,\text{ax}}{(p \to r) \vee (q \to r), p, q, p \to r \vdash r}\,\to_e \qquad \dfrac{\text{Idem}}{(p \to r) \vee (q \to r), p, q, q \to r \vdash r}}{(p \to r) \vee (q \to r), p, q \vdash r}\,\vee_e}{(p \to r) \vee (q \to r), p \vdash q \to r}\,\to_i}{(p \to r) \vee (q \to r) \vdash p \to (q \to r)}\,\to_i$$

4.4 Distributivités d'implications

4.4.1 $p \to (q \wedge r)$ et $(p \to q) \wedge (p \to r)$

4.4.1.1 $p \to (q \wedge r) \vdash (p \to q) \wedge (p \to r)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \to (q \wedge r), p \vdash p \to (q \wedge r)}\,\text{ax} \qquad \dfrac{}{p \to (q \wedge r), p \vdash p}\,\text{ax}}{p \to (q \wedge r), p \vdash q \wedge r}\,\to_e}{p \to (q \wedge r), p \vdash q}\,\wedge_e}{p \to (q \wedge r) \vdash p \to q}\,\to_i \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{p \to (q \wedge r), p \vdash p \to (q \wedge r)}\,\text{ax} \qquad \dfrac{}{p \to (q \wedge r), p \vdash p}\,\text{ax}}{p \to (q \wedge r), p \vdash q \wedge r}\,\to_e}{p \to (q \wedge r), p \vdash r}\,\wedge_e}{p \to (q \wedge r) \vdash p \to r}\,\to_i}{p \to (q \wedge r) \vdash (p \to q) \wedge (p \to r)}\,\wedge_i$$

4.4.1.2 $(p \to q) \wedge (p \to r) \vdash p \to (q \wedge r)$

On note $\Gamma = (p \to q) \wedge (p \to r)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, p \vdash p \to q}\,\text{ax} \qquad \dfrac{}{\Gamma, p \vdash p}\,\text{ax}}{(p \to q) \wedge (p \to r), p \vdash q}\,\to_e \qquad \dfrac{\dfrac{}{\Gamma, p \vdash p \to r}\,\text{ax} \qquad \dfrac{}{\Gamma, p \vdash p}\,\text{ax}}{(p \to q) \wedge (p \to r), p \vdash r}\,\to_e}{(p \to q) \wedge (p \to r), p \vdash q \wedge r}\,\wedge_i}{p \to q, p \to r \vdash p \to (q \wedge r)}\,\to_i$$

4.4.2 $(p \vee q) \to r$ et $(p \to r) \wedge (q \to r)$

4.4.2.1 $(p \vee q) \to r \vdash (p \to r) \wedge (q \to r)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(p \vee q) \to r, p \vdash (p \vee q) \to r}\,\text{ax} \qquad \dfrac{\dfrac{}{(p \vee q) \to r, p \vdash p}\,\text{ax}}{(p \vee q) \to r, p \vdash p \vee q}\,\vee_i}{(p \vee q) \to r, p \vdash r}\,\to_e}{(p \vee q) \to r \vdash p \to r}\,\to_i \qquad \dfrac{\dfrac{\dfrac{}{(p \vee q) \to r, q \vdash (p \vee q) \to r}\,\text{ax} \qquad \dfrac{\dfrac{}{(p \vee q) \to r, q \vdash q}\,\text{ax}}{(p \vee q) \to r, q \vdash p \vee q}\,\vee_i}{(p \vee q) \to r, q \vdash r}\,\to_e}{(p \vee q) \to r \vdash q \to r}\,\to_i}{(p \vee q) \to r \vdash (p \to r) \wedge (q \to r)}\,\wedge_i$$

4.4.2.2 $p \to r, q \to r \vdash (p \vee q) \to r$

On pose $\Gamma = p \to r, q \to r, p \vee q$.

$$\dfrac{\dfrac{\dfrac{}{p \to r, q \to r, p \vee q \vdash p \vee q}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma, p \vdash p \to r}\,\text{ax} \qquad \dfrac{}{\Gamma, p \vdash p}\,\text{ax}}{p \to r, q \to r, p \vee q, p \vdash r}\,\to_e \qquad \dfrac{\dfrac{}{\Gamma, q \vdash q \to r}\,\text{ax} \qquad \dfrac{}{\Gamma, q \vdash q}\,\text{ax}}{p \to r, q \to r, p \vee q, q \vdash r}\,\to_e}{p \to r, q \to r, p \vee q \vdash r}\,\vee_e}{p \to r, q \to r \vdash (p \vee q) \to r}\,\to_i$$

4.4.3 $(p \to q) \vee (p \to r)$ et $p \to (q \vee r)$

4.4.3.1 $(p \to q) \vee (p \to r) \vdash p \to (q \vee r)$

Ce résultat se déduit en logique classique, voir l'exercice 6.1.10

4.4.3.2 $(p \to q) \vee (p \to r) \vdash p \to (q \vee r)$

On note $\varphi = (p \to q) \vee (p \to r)$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\varphi, p \vdash (p \to q) \vee (p \to r)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\varphi, p, p \to q \vdash p \to q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\varphi, p, p \to q \vdash p$}
\RightLabel{$\to_e$}
\BinaryInfC{$\varphi, p, p \to q \vdash q$}
\RightLabel{$\vee_i$}
\UnaryInfC{$\varphi, p, p \to q \vdash q \vee r$}
\AxiomC{Idem}
\UnaryInfC{$\varphi, p, p \to r \vdash q \vee r$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$\varphi, p \vdash q \vee r$}
\RightLabel{$\to_i$}
\UnaryInfC{$(p \to q) \vee (p \to r) \vdash p \to (q \vee r)$}
\end{prooftree}
$$

4.4.4 $q \to r \vdash (p \to q) \to (p \to r)$

4.4.4.1 $q \to r \vdash (p \to q) \to (p \to r)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$q \to r, p \to q, p \vdash q \to r$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$q \to r, p \to q, p \vdash p \to q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$q \to r, p \to q, p \vdash p$}
\RightLabel{$\to_e$}
\BinaryInfC{$q \to r, p \to q, p \vdash q$}
\RightLabel{$\to_e$}
\BinaryInfC{$q \to r, p \to q, p \vdash r$}
\RightLabel{$\to_i$}
\UnaryInfC{$q \to r, p \to q, p \vdash p \to r$}
\RightLabel{$\to_i$}
\UnaryInfC{$q \to r \vdash (p \to q) \to (p \to r)$}
\end{prooftree}
$$

4.4.4.2 $p \to q, q \to r \vdash p \to r$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p \to q, q \to r, p \vdash q \to r$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p \to q, q \to r, p \vdash p \to q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p \to q, q \to r, p \vdash p$}
\RightLabel{$\to_e$}
\BinaryInfC{$p \to q, q \to r, p \vdash q$}
\RightLabel{$\to_e$}
\BinaryInfC{$p \to q, q \to r, p \vdash r$}
\RightLabel{$\to_i$}
\UnaryInfC{$p \to q, q \to r \vdash p \to r$}
\end{prooftree}
$$

5 Logique intuitionniste

5.1 Usage de l'absurdité intuitionniste

Dans cette section l'usage de l'absurdité intuitionniste ($\bot_e$) est possible.

5.1.1 $\neg\neg p, p \vee \neg p \vdash p$

$$\dfrac{\dfrac{}{\neg\neg p, p \vee \neg p \vdash p \vee \neg p}\,\text{ax} \qquad \dfrac{}{\neg\neg p, p \vee \neg p, p \vdash p}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\neg\neg p, p \vee \neg p, \neg p \vdash \neg p}\,\text{ax} \qquad \dfrac{}{\neg\neg p, p \vee \neg p, \neg p \vdash \neg(\neg p)}\,\text{ax}}{\neg\neg p, p \vee \neg p, \neg p \vdash \bot}\,\neg_e}{\neg\neg p, p \vee \neg p, \neg p \vdash p}\,\bot_e}{\neg\neg p, p \vee \neg p \vdash p}\,\vee_e$$

5.1.2 $\neg p \vdash p \to q$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg p, p \vdash p}\,\text{ax} \qquad \dfrac{}{\neg p, p \vdash \neg p}\,\text{ax}}{\neg p, p \vdash \bot}\,\neg_e}{\neg p, p \vdash q}\,\bot_e}{\neg p \vdash p \to q}\,\to_i$$

5.1.3 $p \vee q, \neg q \vdash p$

$$\dfrac{\dfrac{}{p \vee q, \neg q \vdash p \vee q}\,\text{ax} \qquad \dfrac{}{p \vee q, \neg q, p \vdash p}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{p \vee q, \neg q, q \vdash q}\,\text{ax} \qquad \dfrac{}{p \vee q, \neg q, q \vdash \neg q}\,\text{ax}}{p \vee q, \neg q, q \vdash \bot}\,\neg_e}{p \vee q, \neg q, q \vdash p}\,\bot_e}{p \vee q, \neg q \vdash p}\,\vee_e$$

5.1.4 $(p \vee r) \to (q \vee r), \neg r \vdash p \to q$

On note $\Gamma = (p \vee r) \to (q \vee r), \neg r, p$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (p \vee r) \to (q \vee r)}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma \vdash p}\,\text{ax}}{\Gamma \vdash p \vee r}\,\vee_i}{(p \vee r) \to (q \vee r), \neg r, p \vdash q \vee r}\,\to_e \qquad \dfrac{}{(p \vee r) \to (q \vee r), \neg r, p, q \vdash q}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, r \vdash r}\,\text{ax} \qquad \dfrac{}{\Gamma, r \vdash \neg r}\,\text{ax}}{\Gamma, r \vdash \bot}\,\neg_e}{(p \vee r) \to (q \vee r), \neg r, pr \vdash q}\,\bot_e}{(p \vee r) \to (q \vee r), \neg r, p \vdash q}\,\vee_e}{(p \vee r) \to (q \vee r), \neg r \vdash p \to q}\,\to_i$$

5.1.5 $\neg(p \to q) \vdash q \to p$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(p \to q), q, p \vdash q}\,\text{ax}}{\neg(p \to q), q \vdash p \to q}\,\to_i \qquad \dfrac{}{\neg(p \to q), q \vdash \neg(p \to q)}\,\text{ax}}{\neg(p \to q), q \vdash \bot}\,\neg_e}{\neg(p \to q), q \vdash p}\,\bot_e}{\neg(p \to q) \vdash q \to p}\,\to_i$$

5.2 Déduction de l'absurdité intuitionniste

5.2.1 Une autre règle

On transforme l'exercice 5.1.2 en la règle de création d'implication $\dfrac{\Gamma \vdash \neg p}{\Gamma \vdash p \to q} \to_c$

Dériver ($\bot_e$) en utilisant cette règle. On note $p = \bot \to \bot$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash \bot}{\Gamma, \bot \vdash \bot}\,\text{aff}}{\Gamma \vdash \bot \to \bot}\,\to_i}{\Gamma \vdash p}\,\rho \qquad \dfrac{\dfrac{\dfrac{\Gamma \vdash \bot}{\Gamma, p \vdash \bot}\,\text{aff}}{\Gamma \vdash \neg p}\,\neg_i}{\Gamma \vdash p \to q}\,\to_c}{\Gamma \vdash q}\,\to_e$$

5.2.2 Depuis ($\neg\neg_e$)

Dériver ($\bot_e$) en utilisant ($\neg\neg_e$).

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \bot}{\Gamma, \neg p \vdash \bot}\,\text{aff}}{\Gamma \vdash \neg\neg p}\,\neg_i}{\Gamma \vdash p}\,\neg\neg_e$$

5.2.3 Depuis (raa)

Dériver ($\bot_e$) en utilisant (raa).

$$\dfrac{\dfrac{\Gamma \vdash \bot}{\Gamma, \neg p \vdash \bot}\,\text{aff}}{\Gamma \vdash p}\,\text{raa}$$

6 Logique classique

Dans cette partie, l'usage de (te), (raa), ($\neg\neg_e$) ou ($\bot_e$) est possible.

6.1 Implication et disjonctions

6.1.1 $\neg p \to p \vdash p$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg p \to p, \neg p \vdash \neg p \to p}\,\text{ax} \qquad \dfrac{}{\neg p \to p, \neg p \vdash \neg p}\,\text{ax}}{\neg p \to p, \neg p \vdash p}\,{\to_e} \qquad \dfrac{}{\neg p \to p, \neg p \vdash \neg p}\,\text{ax}}{p \to \neg p, \neg p \vdash \bot}\,{\neg_e}}{p \to \neg p \vdash \neg\neg p}\,{\neg_i}}{p \to \neg p, \vdash p}\,{\neg\neg_e}$$

La forme $p \to \neg p \vdash \neg p$ se démontre en logique minimale, voir 3.1.1.

6.1.2 $p \to q \vdash \neg p \vee q$

$$\dfrac{\dfrac{}{p \to q \vdash p \vee \neg p}\,\text{te} \qquad \dfrac{\dfrac{\dfrac{}{p \to q, p \vdash p \to q}\,\text{ax} \qquad \dfrac{}{p \to q, p \vdash p}\,\text{ax}}{p \to q, p \vdash q}\,{\to_e}}{p \to q, p \vdash \neg p \vee q}\,{\vee_i} \qquad \dfrac{\dfrac{}{p \to q, \neg p \vdash \neg p}\,\text{ax}}{p \to q, \neg p \vdash \neg p \vee q}\,{\vee_i}}{p \to q \vdash \neg p \vee q}\,{\vee_e}$$

6.1.3 $\neg p \vee q \vdash p \to q$

$$\dfrac{\dfrac{}{\neg p \vee q \vdash p \vee \neg p}\,\text{te} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg p \vee q, \neg p, p \vdash p}\,\text{ax} \qquad \dfrac{}{\neg p \vee q, \neg p, p \vdash \neg p}\,\text{ax}}{\neg p \vee q, \neg p, p \vdash \bot}\,{\neg_e}}{\neg p \vee q, \neg p, p \vdash q}\,{\bot_e}}{\neg p \vee q, \neg p \vdash p \to q}\,{\to_i} \qquad \dfrac{\dfrac{}{\neg p \vee q, q, p \vdash q}\,\text{ax}}{\neg p \vee q, q \vdash p \to q}\,{\to_i}}{\neg p \vee q \vdash p \to q}\,{\vee_e}$$

6.1.4 $\neg q \to \neg p \vdash p \to q$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg q \to \neg p, p, \neg q \vdash p}\,\text{ax} \qquad \dfrac{\dfrac{}{\neg q \to \neg p, p, \neg q \vdash \neg q \to \neg p}\,\text{ax} \qquad \dfrac{}{\neg q \to \neg p, p, \neg q \vdash \neg q}\,\text{ax}}{\neg q \to \neg p, p, \neg q \vdash \neg p}\,{\to_e}}{\neg q \to \neg p, p, \neg q \vdash \bot}\,{\neg_e}}{\neg q \to \neg p, p \vdash q}\,\text{raa}}{\neg q \to \neg p \vdash p \to q}\,{\to_i}$$

C'est la réciproque de l'exercice 3.2.1

6.1.5 $q \to r, \neg q \to \neg p \vdash p \to r$

On note $\Gamma = q \to r, \neg q \to \neg p, p$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash q \vee \neg q}\,\text{te} \qquad \dfrac{\dfrac{}{\Gamma, q \vdash q \to r}\,\text{ax} \qquad \dfrac{}{\Gamma, q \vdash q}\,\text{ax}}{\Gamma, q \vdash r}\,{\to_e} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \neg q \vdash p}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma, \neg q \vdash \neg q \to \neg p}\,\text{ax} \qquad \dfrac{}{\Gamma, \neg q \vdash \neg q}\,\text{ax}}{\Gamma, \neg q \vdash \neg p}\,{\to_e}}{\Gamma, \neg q \vdash \bot}\,{\neg_e}}{\Gamma, \neg q \vdash r}\,{\bot_e}}{q \to r, \neg q \to \neg p, p \vdash r}\,{\vee_e}}{q \to r, \neg q \to \neg p \vdash p \to r}\,{\to_i}$$

6.1.6 $p \vee q, \neg q \vee r \vdash p \vee r$

On utilise (te) sous la forme $\dfrac{}{p \vee q, \neg q \vee r \vdash q \vee \neg q}\,\text{te}$.

Lemme 1 On commence par supposer q

$$\dfrac{\dfrac{}{p \vee q, \neg q \vee r, q \vdash \neg q \vee r}\,\text{ax} \quad \dfrac{\dfrac{\dfrac{}{p \vee q, \neg q \vee r, q, \neg q \vdash q}\,\text{ax} \quad \dfrac{}{p \vee q, \neg q \vee r, q, \neg q \vdash \neg q}\,\text{ax}}{p \vee q, \neg q \vee r, q, \neg q \vdash \bot}\,\neg_e}{p \vee q, \neg q \vee r, q, \neg q \vdash r}\,\bot_e \quad \dfrac{}{p \vee q, \neg q \vee r, q, r \vdash r}\,\text{ax}}{\dfrac{p \vee q, \neg q \vee r, q \vdash r}{p \vee q, \neg q \vee r, q \vdash p \vee r}\,\vee_i}\,\vee_e$$

Lemme 2 On suppose maintenant $\neg q$.

$$\dfrac{\dfrac{}{p \vee q, \neg q \vee r, \neg q \vdash p \vee q}\,\text{ax} \quad \dfrac{}{p \vee q, \neg q \vee r, \neg q, p \vdash p}\,\text{ax} \quad \dfrac{\dfrac{\dfrac{}{p \vee q, \neg q \vee r, \neg q, q \vdash q}\,\text{ax} \quad \dfrac{}{p \vee q, \neg q \vee r, \neg q, \vdash \neg q}\,\text{ax}}{p \vee q, \neg q \vee r, \neg q, q \vdash \bot}\,\neg_e}{p \vee q, \neg q \vee r, \neg q, \neg q \vdash p}\,\bot_e}{\dfrac{p \vee q, \neg q \vee r, \neg q \vdash p}{p \vee q, \neg q \vee r, \neg q \vdash p \vee r}\,\vee_i}\,\vee_e$$

Conclusion On peut alors tout réunir.

$$\dfrac{\dfrac{}{p \vee q, \neg q \vee r \vdash q \vee \neg q}\,\text{te} \quad \dfrac{\text{Lemme 1}}{p \vee q, \neg q \vee r, q \vdash p \vee r} \quad \dfrac{\text{Lemme 2}}{p \vee q, \neg q \vee r, \neg q \vdash p \vee r}}{p \vee q, \neg q \vee r \vdash p \vee r}\,\vee_e$$

6.1.7 $\neg(p \wedge q) \vdash \neg p \vee \neg q$

On note $\varphi = \neg(p \wedge q)$.

$$\dfrac{\dfrac{}{\varphi \vdash p \vee \neg p}\,\text{te} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi, p, q \vdash \neg(p \wedge q)}\,\text{ax} \quad \dfrac{\dfrac{}{\varphi, p, q \vdash p}\,\text{ax} \quad \dfrac{}{\varphi, p, q \vdash q}\,\text{ax}}{\varphi, p, q \vdash p \wedge q}\,\wedge_i}{\varphi, p, q \vdash \bot}\,\neg_e}{\varphi, p \vdash \neg q}\,\neg_i}{\varphi, p \vdash \neg p \vee \neg q}\,\vee_i \quad \dfrac{\dfrac{}{\varphi, \neg p \vdash \neg p}\,\text{ax}}{\varphi, \neg p \vdash \neg p \vee \neg q}\,\vee_i}{\neg(p \wedge q) \vdash \neg p \vee \neg q}\,\vee_e$$

C'est la réciproque de l'exercice 4.2.2.2

6.1.8 $(p \wedge q) \to r \vdash (p \to r) \vee (q \to r)$

On note $\varphi = (p \wedge q) \to r$.

$$\dfrac{\dfrac{}{\varphi \vdash p \vee \neg p}\,\text{te} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi, p, q \vdash (p \wedge q) \to r}\,\text{ax} \quad \dfrac{\dfrac{}{\varphi, p, q \vdash p}\,\text{ax} \quad \dfrac{}{\varphi, p, q \vdash p}\,\text{ax}}{\varphi, p, q \vdash p \wedge q}\,\wedge_i}{\varphi, p, q \vdash r}\,\to_e}{\varphi, p \vdash q \to r}\,\to_i}{\varphi, p \vdash (p \to r) \vee (q \to r)}\,\vee_i \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi, \neg p, p \vdash p}\,\text{ax} \quad \dfrac{}{\varphi, \neg p, p \vdash \neg p}\,\text{ax}}{\varphi, \neg p, p \vdash \bot}\,\neg_e}{\varphi, \neg p, p \vdash r}\,\bot_e}{\varphi, \neg p \vdash p \to r}\,\to_i}{\varphi, \neg p \vdash (p \to r) \vee (q \to r)}\,\vee_i}{(p \wedge q) \to r \vdash (p \to r) \vee (q \to r)}\,\vee_e$$

C'est la réciproque de l'exercice 4.3.2.2

6.1.9 $p \to (q \to r) \vdash (p \to r) \vee (q \to r)$

On note $\varphi = p \to (q \to r)$.

$$\dfrac{\dfrac{}{p \to (q \to r) \vdash p \vee \neg p}\,\text{te} \qquad \dfrac{\dfrac{\dfrac{}{\varphi, p \vdash p \to (q \to r)}\,\text{ax} \quad \dfrac{}{\varphi, p \vdash p}\,\text{ax}}{p \to (q \to r), p \vdash q \to r}\,\to_e}{p \to (q \to r), p \vdash (p \to r) \vee (q \to r)}\,\vee_i \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi, \neg p, p \vdash p}\,\text{ax} \quad \dfrac{}{\varphi, \neg p, p \vdash \neg p}\,\text{ax}}{p \to (q \to r), \neg p, p \vdash \bot}\,\neg_e}{p \to (q \to r), \neg p, p \vdash r}\,\bot_e}{p \to (q \to r), \neg p \vdash p \to r}\,\to_i}{p \to (q \to r), \neg p \vdash (p \to r) \vee (q \to r)}\,\vee_i}{p \to (q \to r) \vdash (p \to r) \vee (q \to r)}\,\vee_e$$

C'est la réciproque de l'exercice 4.3.3.2

6.1.10 $p \to (q \vee r) \vdash (p \to q) \vee (p \to r)$

Lemme On commence par $p \to (q \vee r), p \vdash (p \to q) \vee (p \to r)$. On note $\varphi = p \to (q \vee r)$.

$$\dfrac{\dfrac{\dfrac{}{\varphi, p \vdash p \to (q \vee r)}\,\text{ax} \quad \dfrac{}{\varphi, p \vdash p}\,\text{ax}}{\varphi, p \vdash q \vee r}\,\to_e \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi, p, q \vdash q}\,\text{ax}}{\varphi, q \vdash p \to q}\,\to_i}{\varphi, q \vdash (p \to q) \vee (p \to r)}\,\vee_i}{\varphi, p, q \vdash (p \to q) \vee (p \to r)}\,\text{aff} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi, p, q \vdash r}\,\text{ax}}{\varphi, r \vdash p \to r}\,\to_i}{\varphi, r \vdash (p \to q) \vee (p \to r)}\,\vee_i}{\varphi, p, q \vdash (p \to q) \vee (p \to r)}\,\text{aff}}{\varphi, p \vdash (p \to q) \vee (p \to r)}\,\vee_e$$

Conclusion On passe alors à la déduction demandée.

$$\dfrac{\dfrac{}{\varphi \vdash p \vee \neg p}\,\text{te} \qquad \dfrac{\text{Lemme}}{\varphi, p \vdash (p \to q) \vee (p \to r)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi, \neg p, p \vdash p}\,\text{ax} \quad \dfrac{}{\varphi, \neg p, p \vdash \neg p}\,\text{ax}}{\varphi, \neg p, p \vdash \bot}\,\neg_e}{\varphi, \neg p, p \vdash q}\,\bot_e}{\varphi, \neg p \vdash p \to q}\,\to_i}{\varphi, \neg p \vdash (p \to q) \vee (p \to r)}\,\vee_i}{p \to (q \vee r) \vdash (p \to q) \vee (p \to r)}\,\vee_e$$

C'est la réciproque de l'exercice 4.4.3.2

6.2 Équivalences des règles

6.2.1 $(\neg\neg_e)$ donne (te)

Dériver (te) en utilisant $(\neg\neg_e)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg(p \vee \neg p), p \vdash p}\,\text{ax}}{\Gamma, \neg(p \vee \neg p), p \vdash p \vee \neg p}\,\vee_i \quad \dfrac{}{\Gamma, \neg(p \vee \neg p), p \vdash \neg(p \vee \neg p)}\,\text{ax}}{\Gamma, \neg(p \vee \neg p), p \vdash \bot}\,\neg_e}{\Gamma, \neg(p \vee \neg p) \vdash \neg p}\,\neg_i}{\Gamma, \neg(p \vee \neg p) \vdash p \vee \neg p}\,\vee_i \quad \dfrac{}{\Gamma, \neg(p \vee \neg p) \vdash \neg(p \vee \neg p)}\,\text{ax}}{\Gamma, \neg(p \vee \neg p) \vdash \bot}\,\neg_e}{\Gamma \vdash \neg\neg(p \vee \neg p)}\,\neg_i}{\Gamma \vdash p \vee \neg p}\,\neg\neg_e$$

6.2.2 (raa) donne (te)

Dériver (te) en utilisant (raa).

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg(p \vee \neg p), p \vdash p$}
\RightLabel{$\vee_i$}
\UnaryInfC{$\Gamma, \neg(p \vee \neg p), p \vdash p \vee \neg p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg(p \vee \neg p), p \vdash \neg(p \vee \neg p)$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \neg(p \vee \neg p), p \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$\Gamma, \neg(p \vee \neg p) \vdash \neg p$}
\RightLabel{$\vee_i$}
\UnaryInfC{$\Gamma, \neg(p \vee \neg p) \vdash p \vee \neg p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg(p \vee \neg p) \vdash \neg(p \vee \neg p)$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \neg(p \vee \neg p) \vdash \bot$}
\RightLabel{raa}
\UnaryInfC{$\Gamma \vdash p \vee \neg p$}
\end{prooftree}
$$

6.2.3 (te) donne ($\neg\neg_e$)

Dériver ($\neg\neg_e$) en utilisant (te) et ($\bot_e$)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{te}
\UnaryInfC{$\Gamma \vdash p \vee \neg p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p \vdash p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg p \vdash \neg p$}
\AxiomC{$\Gamma \vdash \neg\neg p$}
\RightLabel{aff}
\UnaryInfC{$\Gamma, \neg p \vdash \neg(\neg p)$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \neg p \vdash \bot$}
\RightLabel{$\bot_e$}
\UnaryInfC{$\Gamma, \neg p \vdash p$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$\Gamma \vdash p$}
\end{prooftree}
$$

6.2.4 (raa) donne ($\neg\neg_e$)

Dériver ($\neg\neg_e$) en utilisant (raa).

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg p \vdash \neg p$}
\AxiomC{$\Gamma \vdash \neg\neg p$}
\RightLabel{aff}
\UnaryInfC{$\Gamma, \neg p \vdash \neg(\neg p)$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \neg p \vdash \bot$}
\RightLabel{raa}
\UnaryInfC{$\Gamma \vdash p$}
\end{prooftree}
$$

6.2.5 (te) donne (raa)

Dériver (raa) en utilisant (te) et ($\bot_e$)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{te}
\UnaryInfC{$\Gamma \vdash p \vee \neg p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p \vdash p$}
\AxiomC{$\Gamma, \neg p \vdash \bot$}
\RightLabel{$\bot_e$}
\UnaryInfC{$\Gamma, \neg p \vdash p$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$\Gamma \vdash p$}
\end{prooftree}
$$

6.2.6 ($\neg\neg_e$) donne (raa)

Dériver (raa) en utilisant ($\neg\neg_e$).

$$
\begin{prooftree}
\AxiomC{$\Gamma, \neg p \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$\Gamma \vdash \neg\neg p$}
\RightLabel{$\neg\neg_e$}
\UnaryInfC{$\Gamma \vdash p$}
\end{prooftree}
$$

7 Logique du premier ordre

7.1 Divers

7.1.1 $\forall x\varphi \vdash \exists x\varphi$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x\varphi \vdash \forall x\varphi$}
\RightLabel{$\forall_e$}
\UnaryInfC{$\forall x\varphi \vdash \varphi[x \leftarrow t]$}
\RightLabel{$\exists_i$}
\UnaryInfC{$\forall x\varphi \vdash \exists x\varphi$}
\end{prooftree}
$$

7.1.2 $\exists t.\forall x\varphi(t,x) \vdash \forall y\exists x\varphi(z,y)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\exists t.\forall x\varphi(t,x) \vdash \exists t.\forall x\varphi(t,x)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x\varphi(t,x) \vdash \forall x\varphi(t,x)$}
\RightLabel{$\forall_e$}
\UnaryInfC{$\forall x\varphi(t,x) \vdash \varphi(t,x)[x \leftarrow y]$}
\RightLabel{ρ}
\UnaryInfC{$\forall x\varphi(t,x) \vdash \varphi(t,y)$}
\RightLabel{ρ}
\UnaryInfC{$\forall x\varphi(t,x) \vdash \varphi(z,y)[z \leftarrow t]$}
\RightLabel{$\exists_i$}
\UnaryInfC{$\forall x\varphi(t,x) \vdash \exists x\varphi(z,y)$}
\RightLabel{$\forall_i$}
\UnaryInfC{$\forall x\varphi(t,x) \vdash \forall y\exists x\varphi(z,y)$}
\RightLabel{aff}
\UnaryInfC{$\exists t.\forall x\varphi(t,x), \forall x\varphi(t,x) \vdash \forall y\exists x\varphi(z,y)$}
\RightLabel{$\exists_e$}
\BinaryInfC{$\exists t.\forall x\varphi(t,x) \vdash \forall y\exists x\varphi(z,y)$}
\end{prooftree}
$$

On peut appliquer $(\forall_i)$ car y n'est pas dans $\forall x\varphi(t,x)$
On peut appliquer $(\exists_e)$ car t n'est pas libre dans $\exists t.\forall x\varphi(t,x)$ ni dans $\forall y\exists x\varphi(z,y)$.

7.1.3 $\forall x\varphi \vdash \neg(\exists x(\neg\varphi))$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x\varphi, \exists x(\neg\varphi) \vdash \exists x(\neg\varphi)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x\varphi, \exists x(\neg\varphi), \neg\varphi \vdash \forall x\varphi$}
\RightLabel{$\forall_e$}
\UnaryInfC{$\forall x\varphi, \exists x(\neg\varphi), \neg\varphi \vdash \varphi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x\varphi, \exists x(\neg\varphi), \neg\varphi \vdash \neg\varphi$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\forall x\varphi, \exists x(\neg\varphi), \neg\varphi \vdash \bot$}
\RightLabel{$\exists_e$}
\BinaryInfC{$\forall x\varphi, \exists x(\neg\varphi) \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$\forall x\varphi \vdash \neg(\exists x(\neg\varphi))$}
\end{prooftree}
$$

On peut éliminer $\exists$ car x n'est pas une variable libre dans $\forall x\varphi$, $\exists x(\neg\varphi)$ ou $\bot$.
La réciproque demande un raisonnement par l'absurde, voir l'exercice 7.4.1.

7.1.4 $\exists x(\neg\varphi) \vdash \neg(\forall x\varphi)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\exists x(\neg\varphi), \forall x\varphi \vdash \exists x(\neg\varphi)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x\varphi \vdash \forall x\varphi$}
\RightLabel{$\forall_e$}
\UnaryInfC{$\forall x\varphi \vdash \varphi$}
\RightLabel{aff}
\UnaryInfC{$\exists x(\neg\varphi), \forall x\varphi, \neg\varphi \vdash \varphi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\exists x(\neg\varphi), \forall x\varphi, \neg\varphi \vdash \neg\varphi$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\exists x(\neg\varphi), \forall x\varphi, \neg\varphi \vdash \bot$}
\RightLabel{$\exists_e$}
\BinaryInfC{$\exists x(\neg\varphi), \forall x\varphi \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$\exists x(\neg\varphi) \vdash \neg(\forall x\varphi)$}
\end{prooftree}
$$

L'usage de $(\exists_e)$ est possible car x n'est pas une variable libre dans $\exists x(\neg\varphi)$, $\forall x\varphi$ ou $\bot$.
La réciproque demande un raisonnement par l'absurde, voir l'exercice 7.4.2.

7.1.5 $\neg(\exists x\varphi) \vdash \forall x(\neg\varphi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x\varphi),\varphi\vdash\varphi}\;\text{ax}}{\neg(\exists x\varphi),\varphi\vdash\exists x\varphi}\;\exists_i \qquad \dfrac{}{\neg(\exists x\varphi),\varphi\vdash\neg(\exists x\varphi)}\;\text{ax}}{\neg(\exists x\varphi),\varphi\vdash\bot}\;\neg_e}{\neg(\exists x\varphi)\vdash\neg\varphi}\;\neg_i}{\neg(\exists x\varphi)\vdash\forall x(\neg\varphi)}\;\forall_i$$

L'utilisation de $(\forall_i)$ st possible car x n'est pas une variable libre de $\neg(\exists x\varphi)$.
La réciproque demande un raisonnement par l'absurde, voir l'exercice 7.4.3.

7.1.6 $\neg\big(\forall x(\neg\varphi)\big) \vdash \exists x\varphi$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\big(\forall x(\neg\varphi)\big),\neg(\exists x\varphi),\varphi\vdash\varphi}\;\text{ax}}{\neg\big(\forall x(\neg\varphi)\big),\neg(\exists x\varphi),\varphi\vdash\exists x\varphi}\;\exists_i \qquad \dfrac{}{\neg\big(\forall x(\neg\varphi)\big),\neg(\exists x\varphi),\varphi\vdash\neg(\exists x\varphi)}\;\text{ax}}{\neg\big(\forall x(\neg\varphi)\big),\neg(\exists x\varphi),\varphi\vdash\bot}\;\neg_e}{\neg\big(\forall x(\neg\varphi)\big),\neg(\exists x\varphi)\vdash\neg\varphi}\;\bot_e}{\neg\big(\forall x(\neg\varphi)\big),\neg(\exists x\varphi)\vdash\forall x(\neg\varphi)}\;\forall_i \qquad \dfrac{}{\neg\big(\forall x(\neg\varphi)\big),\neg(\exists x\varphi)\vdash\neg\big(\forall x(\neg\varphi)\big)}\;\text{ax}}{\dfrac{\neg\big(\forall x(\neg\varphi)\big),\neg(\exists x\varphi)\vdash\bot}{\neg\big(\forall x(\neg\varphi)\big)\vdash\exists x(\neg\varphi)}\;\to_i}\;\neg_e$$

L'usage de $(\forall_i)$ est possible car x n'est pas une variable libre dans $\neg\big(\forall x(\neg\varphi)\big)$ ni dans $\neg(\exists x\varphi)$.
La réciproque demande un raisonnement par l'absurde, voir l'exercice 7.4.4.

7.1.7 $\exists x\varphi(x), \forall x\forall y\big(\varphi(x)\to\psi(y)\big) \vdash \forall y\psi(y)$

On note $\Gamma = \exists x\varphi(x), \forall x\forall y\big(\varphi(x)\to\psi(y)\big), \varphi(x)$.

$$\dfrac{\dfrac{}{\exists x\varphi(x),\forall x\forall y\big(\varphi(x)\to\psi(y)\big)\vdash\exists x\varphi(x)}\;\text{ax} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\vdash\forall x\forall y\big(\varphi(x)\to\psi(y)\big)}\;\text{ax}}{\Gamma\vdash\forall y\big(\varphi(x)\to\psi(y)\big)[x\leftarrow x]}\;\forall_e}{\Gamma\vdash\forall y\big(\varphi(x)\to\psi(y)\big)}\;\rho}{\Gamma\vdash\big(\varphi(x)\to\psi(y)\big)[y\leftarrow y]}\;\forall_e}{\Gamma\vdash\varphi(x)\to\psi(y)}\;\rho \qquad \dfrac{}{\Gamma\vdash\varphi(x)}\;\text{ax}}{\exists x\varphi(x),\forall x\forall y\big(\varphi(x)\to\psi(y)\big),\varphi(x)\vdash\psi(y)}\;\to_e}{\exists x\varphi(x),\forall x\forall y\big(\varphi(x)\to\psi(y)\big),\varphi(x)\vdash\forall y\psi(y)}\;\forall_i}{\exists x\varphi(x),\forall x\forall y\big(\varphi(x)\to\psi(y)\big)\vdash\forall y\psi(y)}\;\exists_e$$

On peut introduire $\forall$ car y n'est pas une variable libre dans $\exists x\varphi(x)$, $\forall x\forall y\big(\varphi(x)\to\psi(y)\big)$ ou $\varphi(x)$.
On peut éliminer $\exists$ car x n'est pas une variable libre dans $\exists x\varphi(x)$, $\forall x\forall y\big(\varphi(x)\to\psi(y)\big)$ ou $\forall y\psi(y)$.

7.2 Distributivité des quantificateurs

7.2.1 $\forall x(\varphi \land \psi)$ et $(\forall x\varphi) \land (\forall x\psi)$

7.2.1.1 $\forall x(\varphi \land \psi) \vdash (\forall x\varphi) \land (\forall x\psi)$

$$\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{\forall x(\varphi \land \psi) \vdash \forall x(\varphi \land \psi)}
\RightLabel{\forall_e}
\UnaryInfC{\forall x(\varphi \land \psi) \vdash \varphi \land \psi}
\RightLabel{\land_e}
\UnaryInfC{\forall x(\varphi \land \psi) \vdash \varphi}
\RightLabel{\forall_i}
\UnaryInfC{\forall x(\varphi \land \psi) \vdash \forall x\varphi}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{\forall x(\varphi \land \psi) \vdash \forall x(\varphi \land \psi)}
\RightLabel{\forall_e}
\UnaryInfC{\forall x(\varphi \land \psi) \vdash \varphi \land \psi}
\RightLabel{\land_e}
\UnaryInfC{\forall x(\varphi \land \psi) \vdash \psi}
\RightLabel{\forall_i}
\UnaryInfC{\forall x(\varphi \land \psi) \vdash \forall x\psi}
\RightLabel{\land_i}
\BinaryInfC{\forall x(\varphi \land \psi) \vdash (\forall x\varphi) \land (\forall x\psi)}
\end{prooftree}$$

Dans les deux cas $(\forall_i)$ est valide car x n'est pas une variable libre de $\forall x(\varphi \land \psi)$.

7.2.1.2 $(\forall x\varphi) \land (\forall x\psi) \vdash \forall x(\varphi \land \psi)$

$$\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{(\forall x\varphi) \land (\forall x\psi) \vdash (\forall x\varphi) \land (\forall x\psi)}
\RightLabel{\land_e}
\UnaryInfC{(\forall x\varphi) \land (\forall x\psi) \vdash \forall x\varphi}
\RightLabel{\forall_e}
\UnaryInfC{(\forall x\varphi) \land (\forall x\psi) \vdash \varphi}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{(\forall x\varphi) \land (\forall x\psi) \vdash (\forall x\varphi) \land (\forall x\psi)}
\RightLabel{\land_e}
\UnaryInfC{(\forall x\varphi) \land (\forall x\psi) \vdash \forall x\psi}
\RightLabel{\forall_e}
\UnaryInfC{(\forall x\varphi) \land (\forall x\psi) \vdash \psi}
\RightLabel{\land_i}
\BinaryInfC{(\forall x\varphi) \land (\forall x\psi) \vdash \varphi \land \psi}
\RightLabel{\forall_i}
\UnaryInfC{(\forall x\varphi) \land (\forall x\psi) \vdash \forall x(\varphi \land \psi)}
\end{prooftree}$$

$(\forall_i)$ est valide car x n'est pas une variable libre de $(\forall x\varphi) \land (\forall x\psi)$.

7.2.2 $\exists x(\varphi \land \psi) \vdash \exists x\varphi \land \exists x\psi$

$$\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{\exists x(\varphi \land \psi) \vdash \exists x(\varphi \land \psi)}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{\exists x(\varphi \land \psi), \varphi \land \psi \vdash \varphi \land \psi}
\RightLabel{\land_e}
\UnaryInfC{\exists x(\varphi \land \psi), \varphi \land \psi \vdash \varphi}
\RightLabel{\exists_i}
\UnaryInfC{\exists x(\varphi \land \psi), \varphi \land \psi \vdash \exists x\varphi}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{\exists x(\varphi \land \psi), \varphi \land \psi \vdash \varphi \land \psi}
\RightLabel{\land_e}
\UnaryInfC{\exists x(\varphi \land \psi), \varphi \land \psi \vdash \psi}
\RightLabel{\exists_i}
\UnaryInfC{\exists x(\varphi \land \psi), \varphi \land \psi \vdash \exists x\psi}
\RightLabel{\land_i}
\BinaryInfC{\exists x(\varphi \land \psi), \varphi \land \psi \vdash \exists x\varphi \land \exists x\psi}
\RightLabel{\exists_e}
\BinaryInfC{\exists x(\varphi \land \psi) \vdash \exists x\varphi \land \exists x\psi}
\end{prooftree}$$

L'usage de $(\exists_e)$ est possible car x n'est pas une variable libre dans $\exists x(\varphi \land \psi)$ ni dans $\exists x\varphi \land \exists x\psi$
La réciproque n'est pas vraie, voir l'exercice 7.3.2.2

7.2.3 $(\forall x\varphi) \lor (\forall x\psi) \vdash \forall x(\varphi \lor \psi)$

$$\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{(\forall x\varphi) \lor (\forall x\psi) \vdash (\forall x\varphi) \lor (\forall x\psi)}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{(\forall x\varphi) \lor (\forall x\psi), \forall x\varphi \vdash \forall x\varphi}
\RightLabel{\forall_e}
\UnaryInfC{(\forall x\varphi) \lor (\forall x\psi), \forall x\varphi \vdash \varphi}
\RightLabel{\lor_i}
\UnaryInfC{(\forall x\varphi) \lor (\forall x\psi), \forall x\varphi \vdash \varphi \lor \psi}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{(\forall x\varphi) \lor (\forall x\psi), \forall x\psi \vdash \forall x\psi}
\RightLabel{\forall_e}
\UnaryInfC{(\forall x\varphi) \lor (\forall x\psi), \forall x\psi \vdash \psi}
\RightLabel{\lor_i}
\UnaryInfC{(\forall x\varphi) \lor (\forall x\psi), \forall x\psi \vdash \varphi \lor \psi}
\RightLabel{\lor_e}
\TrinaryInfC{(\forall x\varphi) \lor (\forall x\psi) \vdash \varphi \lor \psi}
\RightLabel{\forall_i}
\UnaryInfC{(\forall x\varphi) \lor (\forall x\psi) \vdash \forall x(\varphi \lor \psi)}
\end{prooftree}$$

$(\forall_i)$ est valide car x n'est pas une variable libre de $(\forall x\varphi) \lor (\forall x\psi)$.
La réciproque n'est pas vraie, voir l'exercice 7.4.5

7.2.4 $\exists x(\varphi \vee \psi)$ **et** $(\exists x\varphi) \vee (\exists x\psi)$

7.2.4.1 $\exists x(\varphi \vee \psi) \vdash (\exists x\varphi) \vee (\exists x\psi)$

On note $\Gamma = \exists x(\varphi \vee \psi), \varphi \vee \psi$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\exists x(\varphi \vee \psi) \vdash \exists x(\varphi \vee \psi)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \varphi \vee \psi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \varphi \vdash \varphi$}
\RightLabel{$\exists_i$}
\UnaryInfC{$\Gamma, \varphi \vdash \exists x\varphi$}
\RightLabel{$\vee_i$}
\UnaryInfC{$\Gamma, \varphi \vdash (\exists x\varphi) \vee \psi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \psi \vdash \psi$}
\RightLabel{$\exists_i$}
\UnaryInfC{$\Gamma, \psi \vdash \exists x\psi$}
\RightLabel{$\vee_i$}
\UnaryInfC{$\Gamma, \psi \vdash (\exists x\varphi) \vee (\exists x\psi)$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$\exists x(\varphi \vee \psi), \varphi \vee \psi \vdash (\exists x\varphi) \vee (\exists x\psi)$}
\RightLabel{$\exists_e$}
\BinaryInfC{$\exists x(\varphi \vee \psi) \vdash (\exists x\varphi) \vee (\exists x\psi)$}
\end{prooftree}
$$

L'usage de $(\exists_e)$ est possible car x n'est pas libre dans $\exists x(\varphi \vee \psi)$ ni dans $(\exists x\varphi) \vee (\exists x\psi)$

7.2.4.2 $(\exists x\varphi) \vee (\exists x\psi) \vdash \exists x(\varphi \vee \psi)$

On note $\Gamma = (\exists x\varphi) \vee (\exists x\psi), \exists x\varphi$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(\exists x\varphi) \vee (\exists x\psi) \vdash (\exists x\varphi) \vee (\exists x\psi)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \exists x\varphi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \varphi \vdash \varphi$}
\RightLabel{$\vee_i$}
\UnaryInfC{$\Gamma, \varphi \vdash \varphi \vee \psi$}
\RightLabel{$\exists_i$}
\UnaryInfC{$\Gamma, \varphi \vdash \exists x(\varphi \vee \psi)$}
\RightLabel{$\exists_e$}
\BinaryInfC{$(\exists x\varphi) \vee (\exists x\psi), \exists x\varphi \vdash \exists x(\varphi \vee \psi)$}
\AxiomC{Idem}
\UnaryInfC{$(\exists x\varphi) \vee (\exists x\psi), \exists x\psi \vdash \exists x(\varphi \vee \psi)$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$(\exists x\varphi) \vee (\exists x\psi) \vdash \exists x(\varphi \vee \psi)$}
\end{prooftree}
$$

$(\exists_e)$ est possible car x n'est pas libre dans $(\exists x\varphi) \vee (\exists x\psi)$, $\exists x\varphi$ ou $\exists x(\varphi \vee \psi)$.

7.2.5 $\forall x(\varphi \to \psi) \vdash (\forall x\varphi) \to (\forall x\psi)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x(\varphi \to \psi), \forall x\varphi \vdash \forall x(\varphi \to \psi)$}
\RightLabel{$\forall_e$}
\UnaryInfC{$\forall x(\varphi \to \psi), \forall x\varphi \vdash \varphi \to \psi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x(\varphi \to \psi), \forall x\varphi \vdash \forall x\varphi$}
\RightLabel{$\forall_e$}
\UnaryInfC{$\forall x(\varphi \to \psi), \forall x\varphi \vdash \varphi$}
\RightLabel{$\to_e$}
\BinaryInfC{$\forall x(\varphi \to \psi), \forall x\varphi \vdash \psi$}
\RightLabel{$\forall_i$}
\UnaryInfC{$\forall x(\varphi \to \psi), \forall x\varphi \vdash \forall x\psi$}
\RightLabel{$\to_i$}
\UnaryInfC{$\forall x(\varphi \to \psi) \vdash \forall x\varphi \to \forall x\psi$}
\end{prooftree}
$$

L'usage de $(\forall_i)$ est possible car x n'est pas une variable libre de $\forall x(\varphi \to \psi)$ ni de $\forall x\varphi$.
Un forme affaiblie et sa réciproque sont faites aux exercices 7.3.5.1 et 7.3.5.2.
On peut voir aussi les exercices 7.3.7.1 et 7.3.7.2.

7.2.6 $\exists x(\varphi \to \psi) \vdash (\forall x\varphi) \to (\exists x\psi)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\exists x(\varphi \to \psi) \vdash \exists x(\varphi \to \psi)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\exists x(\varphi \to \psi), \varphi \to \psi, \forall x\varphi \vdash \varphi \to \psi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\exists x(\varphi \to \psi), \varphi \to \psi, \forall x\varphi \vdash \forall x\varphi$}
\RightLabel{$\forall_e$}
\UnaryInfC{$\exists x(\varphi \to \psi), \varphi \to \psi, \forall x\varphi \vdash \varphi$}
\RightLabel{$\to_e$}
\BinaryInfC{$\exists x(\varphi \to \psi), \varphi \to \psi, \forall x\varphi \vdash \psi$}
\RightLabel{$\exists_i$}
\UnaryInfC{$\exists x(\varphi \to \psi), \varphi \to \psi, \forall x\varphi \vdash \exists x\psi$}
\RightLabel{$\to_i$}
\UnaryInfC{$\exists x(\varphi \to \psi), \varphi \to \psi \vdash (\forall x\varphi) \to (\exists x\psi)$}
\RightLabel{$\exists_e$}
\BinaryInfC{$\exists x(\varphi \to \psi) \vdash (\forall x\varphi) \to (\exists x\psi)$}
\end{prooftree}
$$

$(\exists_e)$ est possible car x n'est pas libre dans $\exists x(\varphi \to \psi)$ ni dans $(\forall x\varphi) \to (\exists x\psi)$.
La réciproque demande un raisonnement par l'absurde, voir l'exercice 7.4.7.
Des formes affaiblies (et leurs réciproques) sont faites aux exercices 7.3.6.2 (7.4.6) et 7.3.8.1 (7.3.8.2)

7.2.7 $\exists x(\varphi \land \psi), \forall x(\psi \to \theta) \vdash \exists x(\varphi \land \theta)$

On note $\Gamma = \exists x(\varphi \land \psi), \forall x(\psi \to \theta), \varphi \land \psi$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\exists x(\varphi \land \psi), \forall x(\psi \to \theta) \vdash \exists x(\varphi \land \psi)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \varphi \land \psi$}
\RightLabel{$\land_e$}
\UnaryInfC{$\Gamma \vdash \varphi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \forall x(\psi \to \theta)$}
\RightLabel{$\forall_e$}
\UnaryInfC{$\Gamma \vdash \psi \to \theta$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \varphi \land \psi$}
\RightLabel{$\land_e$}
\UnaryInfC{$\Gamma \vdash \psi$}
\RightLabel{$\to_e$}
\BinaryInfC{$\Gamma \vdash \theta$}
\RightLabel{$\land_i$}
\BinaryInfC{$\exists x(\varphi \land \psi), \forall x(\psi \to \theta), \psi \to \theta \vdash \varphi \land \theta$}
\RightLabel{$\exists_i$}
\UnaryInfC{$\exists x(\varphi \land \psi), \forall x(\psi \to \theta), \psi \to \theta \vdash \exists x(\varphi \land \theta)$}
\RightLabel{$\exists_e$}
\BinaryInfC{$\exists x(\varphi \land \psi), \forall x(\psi \to \theta) \vdash \exists x(\varphi \land \theta)$}
\end{prooftree}
$$

L'usage de $(\exists_e)$ est possible car x n'est pas une variable libre de $\exists x(\varphi \land \psi)$, $\forall x(\psi \to \theta)$ ou $\exists x(\varphi \land \theta)$.

7.3 Semi-distributivité des quantificateurs

Dans cette partie on suppose que x n'est pas une variable libre dans ψ et on prouve qu'on peut affecter un quantificateur à φ uniquement quand on l'applique à $\varphi \land \psi$, $\varphi \lor \psi$ ou $\psi \to \varphi$.
Dans le cas de $\varphi \to \psi$, le quantificateur est inversé

7.3.1 $\forall x(\varphi \land \psi)$ et $(\forall x \varphi) \land \psi$, x non libre dans ψ

7.3.1.1 $\forall x(\varphi \land \psi) \vdash (\forall x \varphi) \land \psi$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x(\varphi \land \psi) \vdash \forall x(\varphi \land \psi)$}
\RightLabel{$\forall_e$}
\UnaryInfC{$\forall x(\varphi \land \psi) \vdash \varphi \land \psi$}
\RightLabel{$\land_e$}
\UnaryInfC{$\forall x(\varphi \land \psi) \vdash \varphi$}
\RightLabel{$\forall_i$}
\UnaryInfC{$\forall x(\varphi \land \psi) \vdash \forall x \varphi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x(\varphi \land \psi) \vdash \forall x(\varphi \land \psi)$}
\RightLabel{$\forall_e$}
\UnaryInfC{$\forall x(\varphi \land \psi) \vdash \varphi \land \psi$}
\RightLabel{$\land_e$}
\UnaryInfC{$\forall x(\varphi \land \psi) \vdash \psi$}
\RightLabel{$\land_i$}
\BinaryInfC{$\forall x(\varphi \land \psi) \vdash (\forall x \varphi) \land \psi$}
\end{prooftree}
$$

L'usage de $(\forall_i)$ est valide car x n'est pas une variable libre de $\forall x(\varphi \land \psi)$.
On a en fait repris la déduction de 7.2.1.1, simplifiée.

7.3.1.2 $(\forall x \varphi) \land \psi \vdash \forall x(\varphi \land \psi)$, x non libre dans ψ

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(\forall x \varphi) \land \psi \vdash (\forall x \varphi) \land \psi$}
\RightLabel{$\land_e$}
\UnaryInfC{$(\forall x \varphi) \land \psi \vdash \forall x \varphi$}
\RightLabel{$\forall_e$}
\UnaryInfC{$(\forall x \varphi) \land \psi \vdash \varphi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(\forall x \varphi) \land \psi \vdash (\forall x \varphi) \land \psi$}
\RightLabel{$\land_e$}
\UnaryInfC{$(\forall x \varphi) \land \psi \vdash \psi$}
\RightLabel{$\land_i$}
\BinaryInfC{$(\forall x \varphi) \land \psi \vdash \varphi \land \psi$}
\RightLabel{$\forall_i$}
\UnaryInfC{$(\forall x \varphi) \land \psi \vdash \forall x(\varphi \land \psi)$}
\end{prooftree}
$$

$(\forall_i)$ est valide car x n'est pas une variable libre de $(\forall x \varphi) \land \psi$.
On a en fait repris la déduction de 7.2.1.2, simplifiée.

7.3.2 $\exists x(\varphi \land \psi)$ et $(\exists x \varphi) \land \psi$, x non libre dans ψ

7.3.2.1 $\exists x(\varphi \land \psi) \vdash (\exists x \varphi) \land \psi$, x non libre dans ψ

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\exists x(\varphi \land \psi) \vdash \exists x(\varphi \land \psi)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\exists x(\varphi \land \psi), \varphi \land \psi \vdash \varphi \land \psi$}
\RightLabel{$\land_e$}
\UnaryInfC{$\exists x(\varphi \land \psi), \varphi \land \psi \vdash \varphi$}
\RightLabel{$\exists_i$}
\UnaryInfC{$\exists x(\varphi \land \psi), \varphi \land \psi \vdash \exists x \varphi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\exists x(\varphi \land \psi), \varphi \land \psi \vdash \varphi \land \psi$}
\RightLabel{$\land_e$}
\UnaryInfC{$\exists x(\varphi \land \psi), \varphi \land \psi \vdash \psi$}
\RightLabel{$\land_i$}
\BinaryInfC{$\exists x(\varphi \land \psi), \varphi \land \psi \vdash (\exists x \varphi) \land \psi$}
\RightLabel{$\exists_e$}
\BinaryInfC{$\exists x(\varphi \land \psi) \vdash (\exists x \varphi) \land \psi$}
\end{prooftree}
$$

$(\exists_e)$ est valide car x n'est pas une variable libre de $\exists x(\varphi \land \psi)$ ni de $(\exists x \varphi) \land \psi$.

7.3.2.2 $(\exists x\varphi) \wedge \psi \vdash \exists x(\varphi \wedge \psi)$, x non libre dans ψ

$$\dfrac{\dfrac{\dfrac{}{(\exists x\varphi) \wedge \psi, \varphi \vdash (\exists x\varphi) \wedge \psi}\,\text{ax}}{(\exists x\varphi) \wedge \psi \vdash \exists x\varphi}\,\wedge_e \qquad \dfrac{\dfrac{\dfrac{}{(\exists x\varphi) \wedge \psi, \varphi \vdash \varphi}\,\text{ax} \quad \dfrac{\dfrac{}{(\exists x\varphi) \wedge \psi, \varphi \vdash (\exists x\varphi) \wedge \psi}\,\text{ax}}{(\exists x\varphi) \wedge \psi \vdash \psi}\,\wedge_e}{(\exists x\varphi) \wedge \psi, \varphi \vdash \varphi \wedge \psi}\,\wedge_i}{(\exists x\varphi) \wedge \psi, \varphi \vdash \exists x(\varphi \wedge \psi)}\,\exists_i}{(\exists x\varphi) \wedge \psi \vdash \exists x(\varphi \wedge \psi)}\,\exists_e$$

$(\exists_e)$ est valide car x n'est pas une variable libre de $(\exists x\varphi) \wedge \psi$ ni de $\exists x(\varphi \wedge \psi)$.

7.3.3 $\forall x(\varphi \vee \psi)$ et $(\forall x\varphi) \vee \psi$, x non libre dans ψ

7.3.3.1 $\forall x(\varphi \vee \psi) \vdash (\forall x\varphi) \vee \psi$, x non libre dans ψ

Ce résultat se démontre dans la logique classique, voir 7.4.5

7.3.3.2 $(\forall x\varphi) \vee \psi \vdash \forall x(\varphi \vee \psi)$, x non libre dans ψ

$$\dfrac{\dfrac{\dfrac{}{(\forall x\varphi) \vee \psi \vdash (\forall x\varphi) \vee \psi}\,\text{ax} \quad \dfrac{\dfrac{\dfrac{}{(\forall x\varphi) \vee \psi, \forall x\varphi \vdash \forall x\varphi}\,\text{ax}}{(\forall x\varphi) \vee \psi, \forall x\varphi \vdash \varphi}\,\forall_e}{(\forall x\varphi) \vee \psi, \forall x\varphi \vdash \varphi \vee \psi}\,\vee_i \quad \dfrac{\dfrac{}{(\forall x\varphi) \vee \psi, \psi \vdash \psi}\,\text{ax}}{(\forall x\varphi) \vee \psi, \psi \vdash \varphi \vee \psi}\,\vee_i}{(\forall x\varphi) \vee \psi \vdash \varphi \vee \psi}\,\vee_e}{(\forall x\varphi) \vee \psi \vdash \forall x(\varphi \vee \psi)}\,\forall_i$$

$(\forall_i)$ est valide car x n'est pas une variable libre de $(\forall x\varphi) \vee \psi$.

7.3.4 $\exists x(\varphi \vee \psi)$ et $(\exists x\varphi) \vee \psi$, x non libre dans ψ

7.3.4.1 $\exists x(\varphi \vee \psi) \vdash (\exists x\varphi) \vee \psi$, x non libre dans ψ

On note $\Gamma = \exists x(\varphi \vee \psi), \varphi \vee \psi$.

$$\dfrac{\dfrac{}{\exists x(\varphi \vee \psi) \vdash \exists x(\varphi \vee \psi)}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma \vdash \varphi \vee \psi}\,\text{ax} \quad \dfrac{\dfrac{\dfrac{}{\Gamma, \varphi \vdash \varphi}\,\text{ax}}{\Gamma, \varphi \vdash \exists x\varphi}\,\exists_i}{\Gamma, \varphi \vdash (\exists x\varphi) \vee \psi}\,\vee_i \quad \dfrac{\dfrac{}{\Gamma, \psi \vdash \psi}\,\text{ax}}{\Gamma, \psi \vdash (\exists x\varphi) \vee \psi}\,\vee_i}{\exists x(\varphi \vee \psi), \varphi \vee \psi \vdash (\exists x\varphi) \vee \psi}\,\vee_e}{\exists x(\varphi \vee \psi) \vdash (\exists x\varphi) \vee \psi}\,\exists_e$$

L'utilisation de $(\exists_i)$ est possible car x n'est pas libre dans $\exists x(\varphi \vee \psi)$ ni dans $(\exists x\varphi) \vee \psi$.
On a en fait repris la déduction de 7.2.4.1, simplifiée.

7.3.4.2 $(\exists x\varphi) \vee \psi \vdash \exists x(\varphi \vee \psi)$, x non libre dans ψ

On note $\Gamma = (\exists x\varphi) \vee \psi, \exists x\varphi$.

$$\dfrac{\dfrac{}{(\exists x\varphi) \vee \psi \vdash (\exists x\varphi) \vee \psi}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma \vdash \exists x\varphi}\,\text{ax} \quad \dfrac{\dfrac{\dfrac{}{\Gamma, \varphi \vdash \varphi}\,\text{ax}}{\Gamma, \varphi \vdash \varphi \vee \psi}\,\vee_i}{\Gamma, \varphi \vdash \exists x(\varphi \vee \psi)}\,\exists_i}{(\exists x\varphi) \vee \psi, \exists x\varphi \vdash \exists x(\varphi \vee \psi)}\,\exists_e \quad \dfrac{\dfrac{\dfrac{}{(\exists x\varphi) \vee \psi, \psi \vdash \psi}\,\text{ax}}{(\exists x\varphi) \vee \psi, \psi \vdash \varphi \vee \psi}\,\vee_i}{(\exists x\varphi) \vee \psi, \psi \vdash \exists x(\varphi \vee \psi)}\,\exists_i}{(\exists x\varphi) \vee \psi \vdash \exists x(\varphi \vee \psi)}\,\vee_e$$

$(\exists_e)$ est possible car x n'est pas libre dans $(\exists x\varphi) \vee \psi$, $\exists x\varphi$ ou $\exists x(\varphi \vee \psi)$.
On a en fait repris la déduction de 7.2.4.2, simplifiée.

7.3.5 $\forall x(\psi \to \varphi)$ et $\psi \to (\forall x\varphi)$, x non libre dans ψ

7.3.5.1 $\forall x(\psi \to \varphi) \vdash \psi \to (\forall x\varphi)$, x non libre dans ψ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x(\psi \to \varphi), \psi \vdash \forall x(\psi \to \varphi)}\,\text{ax}}{\forall x(\psi \to \varphi), \psi \vdash \psi \to \varphi}\,\forall_e \qquad \dfrac{}{\forall x(\psi \to \varphi), \psi \vdash \psi}\,\text{ax}}{\forall x(\psi \to \varphi), \psi \vdash \varphi}\,\to_e}{\forall x(\psi \to \varphi), \psi \vdash \forall x\varphi}\,\forall_i}{\forall x(\psi \to \varphi) \vdash \psi \to (\forall x\varphi)}\,\to_i$$

$(\forall_i)$ est possible car x n'est pas une variable libre de $\forall x(\psi \to \varphi)$ ni de ψ.
On a en fait repris la déduction de 7.2.5, simplifiée.

7.3.5.2 $\psi \to (\forall x\varphi) \vdash \forall x(\psi \to \varphi)$, x non libre dans ψ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\psi \to (\forall x\varphi), \psi \vdash \psi \to (\forall x\varphi)}\,\text{ax} \qquad \dfrac{}{\psi \to (\forall x\varphi), \psi \vdash \varphi}\,\text{ax}}{\psi \to (\forall x\varphi), \psi \vdash \forall x\varphi}\,\to_e}{\psi \to (\forall x\varphi), \psi \vdash \varphi}\,\forall_e}{\psi \to (\forall x\varphi) \vdash \psi \to \varphi}\,\to_i}{\psi \to (\forall x\varphi) \vdash \forall x(\psi \to \varphi)}\,\forall_i$$

$(\forall_i)$ est possible car x n'est pas une variable libre de $\psi \to (\forall x\varphi)$.

7.3.6 $\psi \to (\exists x\varphi)$ et $\exists x(\psi \to \varphi)$, x non libre dans ψ

7.3.6.1 $\psi \to (\exists x\varphi) \vdash \exists x(\psi \to \varphi)$, x non libre dans ψ

Ce résultat se démontre dans la logique classique, voir 7.4.6

7.3.6.2 $\exists x(\psi \to \varphi) \vdash \psi \to (\exists x\varphi)$, x non libre dans ψ

$$\dfrac{\dfrac{}{\exists x(\psi \to \varphi) \vdash \exists x(\psi \to \varphi)}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\exists x(\psi \to \varphi), \psi \to \varphi, \psi \vdash \psi \to \varphi}\,\text{ax} \qquad \dfrac{}{\exists x(\psi \to \varphi), \psi \to \varphi, \psi \vdash \varphi}\,\text{ax}}{\exists x(\psi \to \varphi), \psi \to \varphi, \psi \vdash \varphi}\,\to_e}{\exists x(\psi \to \varphi), \psi \to \varphi, \psi \vdash \exists x\varphi}\,\exists_i}{\exists x(\psi \to \varphi), \psi \to \varphi \vdash \psi \to (\exists x\psi)}\,\to_i}{\exists x(\psi \to \varphi) \vdash \psi \to (\exists x\psi)}\,\exists_e$$

$(\exists_e)$ est possible car x n'est pas libre dans $\exists x(\psi \to \varphi)$ ni dans $(\psi) \to (\exists x\psi)$.
On a en fait repris la déduction de 7.2.6, simplifiée.

7.3.7 $\forall x(\varphi \to \psi)$ et $(\exists x\varphi) \to \psi$, x non libre dans ψ

7.3.7.1 $\forall x(\varphi \to \psi) \vdash (\exists x\varphi) \to \psi$, x non libre dans ψ

$$\dfrac{\dfrac{\dfrac{}{\forall x(\varphi \to \psi), \exists x\varphi \vdash \exists x\varphi}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\forall x(\varphi \to \psi), \exists x\varphi, \varphi \vdash \forall x(\varphi \to \psi)}\,\text{ax}}{\forall x(\varphi \to \psi), \exists x\varphi, \varphi \vdash \varphi \to \psi}\,\forall_e \qquad \dfrac{}{\forall x(\varphi \to \psi), \exists x\varphi, \varphi \vdash \varphi}\,\text{ax}}{\forall x(\varphi \to \psi), \exists x\varphi, \varphi \vdash \psi}\,\to_e}{\forall x(\varphi \to \psi), \exists x\varphi \vdash \psi}\,\exists_e}{\forall x(\varphi \to \psi) \vdash \exists x\varphi \to \psi}\,\to_i$$

L'utilisation de $(\exists_e)$ est valide car x n'est pas une variable libre de $\forall x(\varphi \to \psi)$, $\exists x\varphi$ ou ψ.

7.3.7.2 $(\exists x\varphi) \to \psi \vdash \forall x(\varphi \to \psi)$, x non libre dans ψ

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(\exists x\varphi) \to \psi, \varphi \vdash (\exists x\varphi) \to \psi}\,\text{ax} \qquad \dfrac{\dfrac{}{(\exists x\varphi) \to \psi, \varphi \vdash \varphi}\,\text{ax}}{(\exists x\varphi) \to \psi, \varphi \vdash \exists x\varphi}\,\exists_i}{(\exists x\varphi) \to \psi, \varphi \vdash \psi}\,\to_e}{(\exists x\varphi) \to \psi \vdash \varphi \to \psi}\,\to_i}{(\exists x\varphi) \to \psi \vdash \forall x(\varphi \to \psi)}\,\forall_i$$

L'utilisation de $(\forall_i)$ est valide car x n'est pas une variable libre de $(\exists x\varphi) \to \psi$.

7.3.8 $\exists x(\varphi \to \psi)$ et $(\forall x\varphi) \to \psi$, x non libre dans ψ

7.3.8.1 $\exists x(\varphi \to \psi) \vdash (\forall x\varphi) \to \psi$, x non libre dans ψ

$$\dfrac{\dfrac{}{\exists x(\varphi \to \psi) \vdash \exists x(\varphi \to \psi)}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\exists x(\varphi \to \psi), \varphi \to \psi, \forall x\varphi \vdash \varphi \to \psi}\,\text{ax} \qquad \dfrac{\dfrac{}{\exists x(\varphi \to \psi), \varphi \to \psi, \forall x\varphi \vdash (\forall x\varphi)}\,\text{ax}}{\exists x(\varphi \to \psi), \varphi \to \psi, \forall x\varphi \vdash \varphi[x \leftarrow x]}\,\forall_e}{\exists x(\varphi \to \psi), \varphi \to \psi, \forall x\varphi \vdash \psi}\,\to_e}{\exists x(\varphi \to \psi), \varphi \to \psi \vdash (\forall x\varphi) \to \psi}\,\to_i}{\exists x(\varphi \to \psi) \vdash (\forall x\varphi) \to \psi}\,\exists_e$$

L'utilisation de $(\exists_e)$ est valide car x n'est pas une variable libre de $\exists x(\varphi \to \psi)$ ni de $(\forall x\varphi) \to \psi$.

7.3.8.2 $(\forall x\varphi) \to \psi \vdash \exists x(\varphi \to \psi)$, x non libre dans ψ

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(\exists x\varphi) \to \psi, \varphi \vdash (\exists x\varphi) \to \psi}\,\text{ax} \qquad \dfrac{\dfrac{}{(\exists x\varphi) \to \psi, \varphi \vdash \varphi}\,\text{ax}}{(\exists x\varphi) \to \psi, \varphi \vdash \exists x\varphi}\,\exists_i}{(\exists x\varphi) \to \psi, \varphi \vdash \psi}\,\to_e}{(\exists x\varphi) \to \psi \vdash \varphi \to \psi}\,\to_i}{(\exists x\varphi) \to \psi \vdash \forall x(\varphi \to \psi)}\,\forall_i$$

L'usage de $(\forall_i)$ est valide car x n'est pas une variable libre de $(\exists x\varphi) \to \psi$.

7.4 Logique classique du premier ordre

Dans cette partie, l'usage de (te), (raa), ($\neg\neg_e$) ou ($\bot_e$) est possible.

7.4.1 $\neg\big(\exists x(\neg\varphi)\big) \vdash \forall x\varphi$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\big(\exists x(\neg\varphi)\big), \neg\varphi \vdash \neg\varphi}\,\text{ax}}{\neg\big(\exists x(\neg\varphi)\big), \neg\varphi \vdash \exists x(\neg\varphi)}\,\exists_i \qquad \dfrac{}{\neg\big(\exists x(\neg\varphi)\big), \neg\varphi \vdash \neg\big(\exists x(\neg\varphi)\big),}\,\text{ax}}{\neg\big(\exists x(\neg\varphi)\big), \neg\varphi \vdash \bot}\,\neg_e}{\neg\big(\exists x(\neg\varphi)\big) \vdash \varphi}\,\text{raa}}{\neg\big(\exists x(\neg\varphi)\big) \vdash \forall x\varphi}\,\forall_i$$

On peut utiliser ($\forall_i$) car x n'est pas une variable libre de $\neg(\exists x(\neg\varphi))$.
C'est la réciproque de l'exercice 7.1.3

7.4.2 $\neg(\forall x\varphi) \vdash \exists x(\neg\varphi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\forall x\varphi), \neg\big(\exists x(\neg\varphi)\big), \neg\varphi \vdash \neg\varphi}\,\text{ax}}{\neg(\forall x\varphi), \neg\big(\exists x(\neg\varphi)\big), \neg\varphi \vdash \exists x(\neg\varphi)}\,\exists_i \qquad \dfrac{}{\neg(\forall x\varphi), \neg\big(\exists x(\neg\varphi)\big), \neg\varphi \vdash \neg\big(\exists x(\neg\varphi)\big)}\,\text{ax}}{\neg(\forall x\varphi), \neg\big(\exists x(\neg\varphi)\big), \neg\varphi \vdash \bot}\,\neg_e}{\neg(\forall x\varphi), \neg\big(\exists x(\neg\varphi)\big) \vdash \varphi}\,\text{raa}}{\neg(\forall x\varphi), \neg\big(\exists x(\neg\varphi)\big) \vdash \forall x\varphi}\,\forall_i \qquad \dfrac{}{\neg(\forall x\varphi), \neg\big(\exists x(\neg\varphi)\big) \vdash \neg(\forall x\varphi)}\,\text{ax}}{\neg(\forall x\varphi), \neg\big(\exists x(\neg\varphi)\big) \vdash \bot}\,\neg_e}{\neg(\forall x\varphi) \vdash \exists x(\neg\varphi)}\,\to_i$$

L'usage de ($\forall_i$) est possible car x n'est pas une variable libre dans $\neg(\forall x\varphi)$ ni dans $\neg\big(\exists x(\neg\varphi)\big)$.
C'est la réciproque de l'exercice 7.1.4.

7.4.3 $\forall x(\neg\varphi) \vdash \neg(\exists x\varphi)$

$$\dfrac{\dfrac{\dfrac{}{\forall x(\neg\varphi), \exists x\varphi \vdash \exists x\varphi}\,\text{ax} \qquad \dfrac{\dfrac{}{\forall x(\neg\varphi), \exists x\varphi, \varphi \vdash \varphi}\,\text{ax} \qquad \dfrac{\dfrac{}{\forall x(\neg\varphi), \exists x\varphi, \varphi \vdash \forall x(\neg\varphi)}\,\text{ax}}{\forall x(\neg\varphi), \exists x\varphi, \varphi \vdash \neg\varphi}\,\forall_e}{\forall x(\neg\varphi), \exists x\varphi, \varphi \vdash \bot}\,\neg_e}{\forall x(\neg\varphi), \exists x\varphi \vdash \bot}\,\exists_e}{\forall x(\neg\varphi) \vdash \neg(\exists x\varphi)}\,\text{raa}$$

C'est la réciproque de l'exercice 7.1.5.

7.4.4 $\exists x\varphi \vdash \neg\big(\forall x(\neg\varphi)\big)$

$$\dfrac{\dfrac{\dfrac{}{\exists x\varphi, \forall x(\neg\varphi) \vdash \exists x\varphi}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\exists x\varphi, \forall x(\neg\varphi), \varphi \vdash \forall x(\neg\varphi)}\,\text{ax}}{\exists x\varphi, \forall x(\neg\varphi), \varphi \vdash \neg\varphi}\,\forall_e \qquad \dfrac{}{\exists x\varphi, \forall x(\neg\varphi), \varphi \vdash \varphi}\,\text{ax}}{\exists x(\neg\varphi), \forall x\varphi, \neg\varphi \vdash \bot}\,\neg_e}{\exists x\varphi, \forall x(\neg\varphi) \vdash \bot}\,\exists_e}{\exists x\varphi \vdash \neg\big(\forall x(\neg\varphi)\big)}\,\text{raa}$$

L'usage de ($\exists_e$) est possible car x n'est pas une variable libre dans $\exists x\varphi$, $\forall x(\neg\varphi)$ ou $\bot$.
C'est la réciproque de l'exercice 7.1.6

7.4.5 $\forall x(\varphi \vee \psi) \vdash (\forall x\varphi) \vee \psi$, x non libre dans ψ

Lemme On commence par $\forall x(\varphi \vee \psi), \neg\psi \vdash (\forall x\varphi) \vee \psi$; on note $\theta = \forall x(\varphi \vee \psi)$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\theta, \neg\psi \vdash \forall x(\varphi \vee \psi)$}
\RightLabel{$\forall_e$}
\UnaryInfC{$\theta, \neg\psi \vdash \varphi \vee \psi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\theta, \neg\psi, \varphi \vdash \varphi$}
\RightLabel{$\forall_i$}
\UnaryInfC{$\theta, \neg\psi, \varphi \vdash \forall x\varphi$}
\RightLabel{$\vee_i$}
\UnaryInfC{$\theta, \neg\psi, \varphi \vdash (\forall x\varphi) \vee \psi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\theta, \neg\psi, \psi \vdash \psi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\theta, \neg\psi, \psi \vdash \neg\psi$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\theta, \neg\psi, \psi \vdash \bot$}
\RightLabel{$\bot_e$}
\UnaryInfC{$\theta, \neg\psi, \psi \vdash (\forall x\varphi) \vee \psi$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$\forall x(\varphi \vee \psi), \neg\psi \vdash (\forall x\varphi) \vee \psi$}
\end{prooftree}
$$

L'utilisation de $(\forall_i)$ est possible car x n'est pas libre dans $\forall x(\varphi \vee \psi)$ ni dans $\neg\psi$.

Conclusion On peut utiliser le tiers-exclu

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{te}
\UnaryInfC{$\theta \vdash \psi \vee \neg\psi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\theta, \psi \vdash \psi$}
\RightLabel{$\vee_i$}
\UnaryInfC{$\theta, \psi \vdash (\forall x\varphi) \vee \psi$}
\AxiomC{Lemme}
\UnaryInfC{$\forall x(\varphi \vee \psi), \neg\psi \vdash (\forall x\varphi) \vee \psi$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$\forall x(\varphi \vee \psi) \vdash (\forall x\varphi) \vee \psi$}
\end{prooftree}
$$

C'est la réciproque de l'exercice 7.3.3.2.

7.4.6 $\psi \to (\exists x\varphi) \vdash \exists x(\psi \to \varphi)$, x non libre dans ψ

Lemme 1 On commence par $\psi \to (\exists x\varphi), \psi \vdash \exists x(\psi \to \varphi)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\psi \to (\exists x\varphi), \psi \vdash \psi \to (\exists x\varphi)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\psi \to (\exists x\varphi), \psi \vdash \psi$}
\RightLabel{$\to_e$}
\BinaryInfC{$\psi \to (\exists x\varphi), \psi \vdash \exists x\varphi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\psi \to (\exists x\varphi), \psi, \varphi \vdash \varphi$}
\RightLabel{$\to_i$}
\UnaryInfC{$\psi \to (\exists x\varphi), \varphi \vdash \psi \to \varphi$}
\RightLabel{aff}
\UnaryInfC{$\psi \to (\exists x\varphi), \varphi, \psi \vdash \psi \to \varphi$}
\RightLabel{$\exists_i$}
\UnaryInfC{$\psi \to (\exists x\varphi), \varphi, \psi \vdash \exists x(\psi \to \varphi)$}
\RightLabel{$\exists_e$}
\BinaryInfC{$\psi \to (\exists x\varphi), \psi \vdash \exists x(\psi \to \varphi)$}
\end{prooftree}
$$

$(\exists_e)$ est possible car x n'est pas une variable libre de $\psi \to (\exists x\varphi)$, ψ ou $\exists x(\psi \to \varphi)$.

Lemme 2 On déduit ensuite $\psi \to (\exists x\varphi), \neg\psi \vdash \exists x(\psi \to \varphi)$; c'est en fait l'analogue de l'exercice 5.1.2.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\psi \to (\exists x\varphi), \neg\psi, \psi \vdash \psi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\psi \to (\exists x\varphi), \neg\psi, \psi \vdash \neg\psi$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\psi \to (\exists x\varphi), \neg\psi, \psi \vdash \bot$}
\RightLabel{$\bot_e$}
\UnaryInfC{$\psi \to (\exists x\varphi), \neg\psi, \psi \vdash \varphi$}
\RightLabel{$\to_i$}
\UnaryInfC{$\psi \to (\exists x\varphi), \neg\psi \vdash \psi \to \varphi$}
\RightLabel{$\exists_i$}
\UnaryInfC{$\psi \to (\exists x\varphi), \neg\psi \vdash \exists x(\psi \to \varphi)$}
\end{prooftree}
$$

Conclusion On utilise (te) avec ψ et sa négation.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{te}
\UnaryInfC{$\psi \to (\exists x\varphi) \vdash \psi \vee \neg\psi$}
\AxiomC{Lemme 1}
\UnaryInfC{$\psi \to (\exists x\varphi), \psi \vdash \exists x(\psi \to \varphi)$}
\AxiomC{Lemme 2}
\UnaryInfC{$\psi \to (\exists x\varphi), \neg\psi \vdash \exists x(\psi \to \varphi)$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$\psi \to (\exists x\varphi) \vdash \exists x(\psi \to \varphi)$}
\end{prooftree}
$$

C'est la réciproque de 7.3.6.2

7.4.7 $(\forall x\varphi) \to (\exists x\psi) \vdash \exists x(\varphi \to \psi)$

On va utiliser (te) avec $\exists x(\neg\varphi)$ et sa négation.

Lemme 1 On commence par prouver $(\forall x\varphi) \to (\exists x\psi), \exists x(\neg\varphi) \vdash \exists x(\varphi \to \psi)$

On note $\Gamma = (\forall x\varphi) \to (\exists x\psi), \exists x(\neg\varphi)$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(\forall x\varphi) \to (\exists x\psi), \exists x(\neg\varphi) \vdash \exists x(\neg\varphi)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg\varphi, \varphi \vdash \varphi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg\varphi, \varphi \vdash \neg\varphi$}
\RightLabel{$\neg_e$}
\BinaryInfC{$(\forall x\varphi) \to (\exists x\psi), \exists x(\neg\varphi), \neg\varphi, \varphi \vdash \bot$}
\RightLabel{$\bot_e$}
\UnaryInfC{$(\forall x\varphi) \to (\exists x\psi), \exists x(\neg\varphi), \neg\varphi, \varphi \vdash \psi$}
\RightLabel{$\to_i$}
\UnaryInfC{$(\forall x\varphi) \to (\exists x\psi), \exists x(\neg\varphi), \neg\varphi \vdash \varphi \to \psi$}
\RightLabel{$\exists_i$}
\UnaryInfC{$(\forall x\varphi) \to (\exists x\psi), \exists x(\neg\varphi), \neg\varphi \vdash \exists x(\varphi \to \psi)$}
\RightLabel{$\exists_e$}
\BinaryInfC{$(\forall x\varphi) \to (\exists x\psi), \exists x(\neg\varphi) \vdash \exists x(\varphi \to \psi)$}
\end{prooftree}
$$

L'élimination de $\exists$ est légitime car x n'est pas libre dans Γ ni dans $\exists x(\varphi \to \psi)$.

Lemme 2 On prouve ensuite $(\forall x\varphi) \to (\exists x\psi), \neg\big(\exists x(\neg\varphi)\big) \vdash \exists x(\varphi \to \psi)$.

On note $\Gamma' = (\forall x\varphi) \to (\exists x\psi), \neg\big(\exists x(\neg\varphi)\big)$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma' \vdash (\forall x\varphi) \to (\exists x\psi)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma', \neg\varphi \vdash \neg\varphi$}
\RightLabel{$\exists_i$}
\UnaryInfC{$\Gamma', \neg\varphi \vdash \exists x(\neg\varphi)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma', \neg\varphi \vdash \neg\big(\exists x(\neg\varphi)\big)$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma', \neg\varphi \vdash \bot$}
\RightLabel{raa}
\UnaryInfC{$\Gamma' \vdash \varphi$}
\RightLabel{$\forall_i$}
\UnaryInfC{$\Gamma' \vdash \forall x\varphi$}
\RightLabel{$\to_e$}
\BinaryInfC{$\Gamma' \vdash \exists x\psi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma', \psi, \varphi \vdash \psi$}
\RightLabel{$\to_i$}
\UnaryInfC{$\Gamma', \psi \vdash \varphi \to \psi$}
\RightLabel{$\exists_i$}
\UnaryInfC{$\Gamma', \psi \vdash \exists x(\varphi \to \psi)$}
\RightLabel{$\exists_e$}
\BinaryInfC{$(\forall x\varphi) \to (\exists x\psi), \neg\big(\exists x(\neg\varphi)\big) \vdash \exists x(\varphi \to \psi)$}
\end{prooftree}
$$

L'élimination de $\forall$ est légitime car x n'est pas libre dans Γ'.

L'élimination de $\exists$ est légitime car x n'est pas libre dans Γ' ni dans $\exists x(\varphi \to \psi)$.

Conclusion On note $\theta = (\forall x\varphi) \to (\exists x\psi)$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{te}
\UnaryInfC{$\theta \vdash \exists x(\neg\varphi) \lor \neg\big(\exists x(\neg\varphi)\big)$}
\AxiomC{Lemme 1}
\UnaryInfC{$\theta, \exists x(\neg\varphi) \vdash \exists x(\varphi \to \psi)$}
\AxiomC{Lemme 2}
\UnaryInfC{$\theta, \neg\big(\exists x(\neg\varphi)\big) \vdash \exists x(\varphi \to \psi)$}
\RightLabel{$\lor_e$}
\TrinaryInfC{$(\forall x\varphi) \to (\exists x\psi) \vdash \exists x(\varphi \to \psi)$}
\end{prooftree}
$$

7.4.8 $\vdash \exists x\big(\varphi(x) \to \forall y \varphi(y)\big)$

On utilisera (te) avec $\exists x\big(\neg\varphi(x)\big)$ et sa négation.

Lemme 1 On commence par $\exists x\big(\neg\varphi(x)\big) \vdash \exists x\big(\varphi(x) \to \forall y \varphi(y)\big)$

$$
\dfrac{\dfrac{}{\exists x\big(\neg\varphi(x)\big) \vdash \exists x\big(\neg\varphi(x)\big)}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\varphi(x), \varphi(x) \vdash \varphi(x)}\,\text{ax} \qquad \dfrac{}{\neg\varphi(x), \varphi(x) \vdash \neg\varphi(x)}\,\text{ax}}{\neg\varphi(x), \varphi(x) \vdash \bot}\,\neg_e}{\neg\varphi(x), \varphi(x) \vdash \forall y \varphi(y)}\,\bot_e}{\exists x\big(\neg\varphi(x)\big), \neg\varphi(x), \varphi(x) \vdash \forall y \varphi(y)}\,\text{aff}}{\exists x\big(\neg\varphi(x)\big), \neg\varphi(x) \vdash \varphi(x) \to \forall y \varphi(y)}\,\to_i}{\exists x\big(\neg\varphi(x)\big), \neg\varphi(x) \vdash \exists x\big(\varphi(x) \to \forall y \varphi(y)\big)}\,\exists_i}{\exists x\big(\neg\varphi(x)\big) \vdash \exists x\big(\varphi(x) \to \forall y \varphi(y)\big)}\,\exists_e
$$

L'élimination de $\exists$ est possible car x n'est pas libre dans $\exists x\big(\neg\varphi(x)\big)$ ni dans $\exists x\big(\varphi(x) \to \forall y \varphi(y)\big)$.

Lemme 2 On prouve ensuite $\neg\exists x\big(\neg\varphi(x)\big) \vdash \exists x\big(\varphi(x) \to \forall y \varphi(y)\big)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\exists x\big(\neg\varphi(x)\big), \varphi(x), \neg\varphi(y) \vdash \neg\varphi(y)}\,\text{ax}}{\neg\exists x\big(\neg\varphi(x)\big), \varphi(x), \neg\varphi(y) \vdash \exists x\big(\neg\varphi(x)\big)}\,\exists_i \qquad \dfrac{}{\neg\exists x\big(\neg\varphi(x)\big), \varphi(x), \neg\varphi(y) \vdash \neg\exists x\big(\neg\varphi(x)\big)}\,\text{ax}}{\neg\exists x\big(\neg\varphi(x)\big), \varphi(x), \neg\varphi(y) \vdash \bot}\,\neg_e}{\neg\exists x\big(\neg\varphi(x)\big), \varphi(x) \vdash \varphi(y)}\,\text{raa}}{\neg\exists x\big(\neg\varphi(x)\big), \varphi(x) \vdash \forall y \varphi(y)}\,\forall_i}{\neg\exists x\big(\neg\varphi(x)\big) \vdash \varphi(x) \to \forall y \varphi(y)}\,\to_i}{\neg\exists x\big(\neg\varphi(x)\big) \vdash \exists x\big(\varphi(x) \to \forall y \varphi(y)\big)}\,\exists_i
$$

L'introduction de $\forall$ est possible car y n'est pas libre dans $\exists x\big(\neg\varphi(x)\big)$ ni dans $\varphi(x)$.

Conclusion On peut alors conclure

$$
\dfrac{\dfrac{}{\vdash \exists x\big(\neg\varphi(x)\big) \vee \neg\exists x\big(\neg\varphi(x)\big)}\,\text{te} \qquad \dfrac{\text{Lemme 1}}{\exists x\big(\neg\varphi(x)\big) \vdash \exists x\big(\varphi(x) \to \forall y \varphi(y)\big)} \qquad \dfrac{\text{Lemme 2}}{\neg\exists x\big(\neg\varphi(x)\big) \vdash \exists x\big(\varphi(x) \to \forall y \varphi(y)\big)}}{\vdash \exists x\big(\varphi(x) \to \forall y \varphi(y)\big)}\,\vee_e
$$